Local Search with Congestion in Complex Communication Networks

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Abstract. We present a formalism that is able to cope with search and congestion simultaneously. This formalism avoids the problem of simulating the dynamics of the search-communication process which turns out to be impracticable, specially close to the congestion point where search costs (time) diverge.

1 Introduction

In recent years, the study of static and dynamical properties of complex networks has received a lot of attention [1,2,3,4,5]. Complex networks appear in such diverse disciplines as sociology, biology, chemistry, physics or computer science. In particular, great effort has been exerted to understand the behavior of technologically based communication networks such as the Internet [6], the World Wide Web [7], or e-mail networks [8,9,10]. One of the common facts behind these structures is the short mean distance between nodes. Furthermore, these short paths can be found with strategies that do not precise a complete knowledge of the pattern of interactions between the nodes. Related to the search problem, when the network is facing a number of simultaneous processes, we find that the network can get collapsed because some of these problems are travelling through the same node, and this raises the problem of congestion. We introduce a formalism that enables to handle these two problems in a common framework by writing the dynamical properties of the search process in terms of the topological properties of the network [11].

First we calculate the average number of steps (search cost) needed to find a certain node in the network given the search algorithm and the topology of the network. The calculation is exact if the search algorithm is Markovian. Next, congestion is introduced assuming that the network is formed by nodes that behave like queues, meaning that are able to deliver a finite number of packets at each time step [12,13,14]. In this context, we are able (i) to calculate explicitly the point at which the arrival rate of packets leads to network collapse, in the

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sense that the average time needed to perform a search becomes unbounded, and (ii) to determine, below the point of collapse, how the average search time depends on the rate at which search process are started. In both cases, the relevant quantities are expressed in terms of the topology of the network and the search algorithm.

2 Search

Following the steps introduced in [11] let us consider a single *information packet* at node i whose destination is node k. The probability for the packet to go from i to a new node j in its next step is p_{ij}^k. In particular, $p_{kj}^k = 0 \; \forall j$ so that the packet is *removed* as soon as it arrives to its destination. This formulation is completely general, and the precise form of p_{ij}^k will depend on the search algorithm that will be discussed later on. In particular, when the search is Markovian, p_{ij}^k does not depend on previous positions of the packet. Using matrix notation we can define the matrices D^k whose elements D_{ij}^k are the average number of steps needed to go from i to j for a packet traveling towards k. D_{ij}^k are such that $d_{ik} = D_{ik}^k$. These matrices are related to the probabilities p_{ij}^k through the following expression [11]

$$D^k = \left[(I - p^k)^{-1}\right]^2 p^k, \tag{1}$$

where I is the identity matrix. In particular, the element D_{ik}^k is the average number of steps needed to get from i to k when using the search algorithm given by the set of matrices p^k. When the search algorithm has global knowledge of the structure of the network and the packets follow minimum paths between nodes, the effective distance will coincide with the topological minimum distance; otherwise, the effective distance between nodes will be, in general, larger than the topological minimum distance.

Finally, the average search cost in the network is

$$\overline{d} = \frac{\sum_{i,k} D_{ik}^k}{S(S-1)}, \tag{2}$$

where S is the number of nodes in the network. This expression allows to calculate exactly the average search cost performing simple matrix algebra. Note that simulation based calculation of this quantity would require, in principle, to generate an infinite amount of packets and let them travel from all possible origins to all possible destinations following all possible paths, which are in general arbitrarily long.

3 Load

When there are many simultaneous search process, the effective distance is not a good measure of performance since, even when the distance is small, accumulation of packets can generate long delays. Rather, the characteristic time, τ,

needed to get from the origin to the destination is the right measure. According to Little's law of queuing theory[12], the characteristic time is proportional to the average total load of the network, $\overline{N}$. In our case, the total load is identified with the total number of floating packets, which is the algebraic sum of the single queues of the nodes. Thus minimizing τ is equivalent to minimizing $\overline{N}$. In the following, we show how to calculate the load of a network using only the p^k matrices as has been done for the case of no congestion.

First, we calculate the average number of times, b_{ij}^k, that a packet generated at i and with destination k passes through j,

$$b^k = \sum_{n=1}^{\infty} \left(p^k\right)^n = (I - p^k)^{-1} p^k. \tag{3}$$

Note that the elements b_{ij}^k are sums of probabilities but are not probabilities themselves.

The *effective* betweenness of node j, B_j, is defined as the sum over all possible origins and destinations of the packets, and represents the total number of packets that would pass through j if one packet would be generated at each node at each time step with destination to any other node:

$$B_j = \sum_{i,k} b_{ij}^k. \tag{4}$$

Again, as in the case of the effective distance, when the search algorithm is able to find the minimum paths between nodes, the effective betweenness will coincide with the *topological* betweenness, β_j, as usually defined [15,16].

4 Search and Congestion

Now consider the following general scenario. In a communication network, each node generates one packet at each time step with probability ρ independently of the rest of the nodes. The destination of each of these packets is randomly fixed at the moment of its creation. On the other hand, the nodes are queues that can store as many packets as needed but can deliver, on average, only a finite number of them at each time step—without lost of generality, we fix this number to 1. For low values of ρ the system reaches a steady state in which the total number of *floating* packets in the network $N(t)$ fluctuates around a finite value. As ρ increases, the system undergoes a continuous phase transition from this *free phase* to a *congested phase* in which $N(t) \propto t$ [14]. Right at the critical point, ρ_c, quantities such as $N(t)$ and the characteristic time diverge [17]. In the free phase, there is no accumulation at any node in the network and the number of packets that arrive to node j is, on average, $\rho B_j/(S-1)$. Therefore, a particular node will collapse when $\rho B_j/(S-1) > 1$ and the critical congestion point of the network will be

$$\rho_c = \frac{S-1}{B^*} \tag{5}$$

where B^* is the maximum effective betweenness in the network, that corresponds to the most central node.

To calculate the time average of the load of the network, $\overline{N}$, it is necessary to establish the behavior of the queues. In the general scenario proposed above, the arrival of packets to a given node j is a Poisson process with mean $\mu_j = \rho B_j/(S-1)$. Regarding the delivery of the packets, consider the following model. For a node j with ν_j packets stored in its queue, each packet jumps to the next node (chosen according to the algorithm defined through the matrices p^k) with probability $1/\nu_j$. In this model, the delivery of packets is also a Poisson process. In such a simple case in which both the arrival and the delivery are Poisson processes, queues are called M/M/1 in the computer science literature and the average size of the queues is given by [12]

$$\langle \nu_j \rangle = \frac{\mu_j}{1-\mu_j} = \frac{\frac{\rho B_j}{S-1}}{1-\frac{\rho B_j}{S-1}}. \tag{6}$$

The average load of the network $\overline{N}$ is

$$\overline{N} = \sum_{j=1}^{S} \langle \nu_j \rangle = \sum_{j=1}^{S} \frac{\frac{\rho B_j}{S-1}}{1-\frac{\rho B_j}{S-1}}. \tag{7}$$

It is straightforward to extend the calculations to other types of queues. For instance, the queues used in [13] are such that one packet is delivered deterministically at each time step. These queues are called M/D/1 and the corresponding expression for the size of the queues is $\langle \nu_j \rangle = \mu_j^2/(1-\mu_j)$. Moreover, it is worth noting that the fact that we are able to map the behavior of the nodes to that of M/M/1 queues implies that any conclusion that we are able to draw will be valid in general for any system of M/M/1 queues, and with slight modifications for other types of queues.

There are two interesting limiting cases of equation (7). When ρ is very small, $\langle \nu_j \rangle \approx \mu_j$ and taking into account that $\sum_j B_j = \sum_{i,k} d_{ik}^k$, one obtains

$$\overline{N} \approx \rho S \overline{d} \qquad \rho \to 0. \tag{8}$$

On the other hand, when ρ approaches ρ_c most of the load of the network comes from the most congested node, and therefore

$$\overline{N} \approx \frac{1}{1-\frac{\rho B^*}{S-1}} \qquad \rho \to \rho_c, \tag{9}$$

where B^* is, as before, the betweenness of the most central node. The last two expressions suggest the following interesting problem: to minimize the load of a network it is necessary to minimize the effective distance between nodes if the amount of packets is small, but it is necessary to minimize the largest effective betweenness of the network if the amount of packets is large. The first is accomplished by a *star-like* centralized network, that is, a network with a few central

nodes and all the others connected to them. Rather, the second is accomplished by a distributed, very decentralized, network in which all the nodes support a similar load. In [11] we checked that those are indeed the optimal structures by means of an extensive generalized simulated annealing [18]. This behavior is common to any system of queues provided that the communication depends only on the sender. In queues M/D/1, for example, equation (8) reads $\overline{N} \approx (\rho S\overline{d})^2$ (thus, minimization of $\overline{N}$ still implies minimization of $\overline{d}$) and equation (9) is unchanged.

5 Limitations of the Calculation and Bounds to Other Models

It is worth noting that there are only two assumptions in the calculations above. The first one has already been mentioned: the movement of the packets needs to be Markovian to define the jump probability matrices p^k. Although this is not strictly true in real communication networks — where packets are not allowed usually to go through a given node more than once — it can be seen as a first approximation [19,14,13]. The second assumption is that the jump probabilities p^k_{ij} do not depend on the congestion state of the network, although communication protocols sometimes try to avoid congested regions, and then $B_j = B_j(\rho)$. However, all the derivations above will still be true in a number of general situations, including situations in which the paths that the packets follow are unique, in which the routing tables are fixed, or situations in which the structure of the network is very homogeneous and thus the congestion of all the nodes is similar.

When these two assumptions are fulfilled the calculations are exact. For example, the calculation of ρ_c using equation (5) coincides exactly (within the simulation error) with simulations of the communication model where the communication only depends on the sender of the packet. Compared to situations in which packets avoid congested regions, equations (5)–(9) correspond to the worst case scenario and thus provide bounds to more realistic scenarios in which the search algorithm interactively avoids congestion. Consider, for example, ρ_c in the model presented in [14], where the communication depends not only on the sender but also on the availability of the receiver. The fact that the packets are sent with higher probability to less congested nodes implies that the flow is better balanced among nodes. Although the assumptions of the present calculation do not apply, one would expect that the value of ρ_c estimated analytically will be a lower bound to the real situation in which load is more balanced.

For an accurate description of the algorithm used in the computer simulations we need to specify the set of matrices p^k_{ij}. We have assumed that the movement of the packets is Markovian, this implies that the probabilities do not depend on the packet history. In a purely local search scenario, where the knowledge that the nodes have about the network structure is bounded, the nodes face the problem of forwarding the packets. In order to quantify this information we introduce the radius of knowledge as a measure the distance along network links that a node can identify the destination of a packet. Thus, when this radius is

zero, packets travel completely at random until they reach its destination, and then we have

$$p_{ij}^k = (1 - \delta_{ik}) \frac{a_{ij}}{\sum_l a_{il}}. \tag{10}$$

where a_{ij} are the elements of the adjacency matrix of the network: $a_{ij} = 1$ if i and j are connected in the network and $a_{ij} = 0$ otherwise. This expression means that packets are uniformly distributed among neighbors unless they have reach their destination. The delta symbol ensures that $p_{kj}^k = 0 \; \forall j$ and the packet *disappears* from the network. When the radius of knowledge is 1, a node is able to recognize one of its neighbors as the destination and then the packet is sent to it; otherwise, the packet is just sent at random to one of the neighbors of the node. The corresponding p^k matrices are given by

$$p_{ij}^k = a_{ik}\delta_{jk} + (1 - a_{ik} - \delta_{ik}) \frac{a_{ij}}{\sum_l a_{il}}. \tag{11}$$

The first term corresponds to i and k being neighbors: then the packet will go to j if and only if $j = k$, i.e. the packet will be sent directly to the destination. The second term corresponds to i and k not being neighbors: in this case, j is chosen at random among the neighbors of i.

We have compared our predictions using a radius of knowledge equal to one with simulations of the communication model introduced in [14], where the communication between the nodes depends on the receiver's state as well, and packets are automatically delivered if a neighbor is its final destination. Figure 1 shows that our model is indeed a lower bound and, moreover, that the analytical estimation provides a good approximation to the simulated value.

This figure also confirms another expected and useful result. For a given size of the network and a given number of links, the most robust networks, that is those with higher ρ_c, are those with better balanced load. For these networks, the effect of avoiding congestion is less important and therefore the analytical estimation turns out to be more accurate.

6 Packet Dynamics

We have performed simulations of the communication model described in the previous sections for the structures shown in [11] to be optimal in the two regimes: for a low load, the star-like centralized one, and for a high load, the distributed one. We note that for a fixed number of nodes and links, the distributed network has a larger value of ρ_c than its centralized counterpart. In the case we are considering of 32 nodes and 128 links, the critical values of ρ_c are 0.256 and 0.162, respectively. Nevertheless, the dynamical behavior near the critical value of the load has to be very similar. In Fig. 1 we have plotted the standard deviation in the number of packets in the most connected nodes and in the total number of packets in the network.

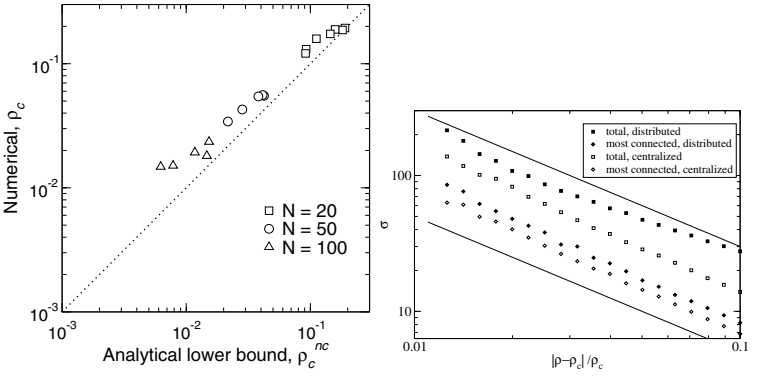

Fig. 1. Left: Comparison between the predictions of equation (5) for ρ_c with betweennesses computed according to (11) and the results obtained for the communication model discussed in [14]. The analytical value is a lower bound to the actual value. To keep the figure simple, we do not show results corresponding to the model discussed in the current work, but the points would lay exactly on the diagonal line, since all the assumptions of the calculation are fulfilled. Right: Standard deviation in (squares) total number of packets and (diamonds) number of packets in the most connected nodes. The straight lines have slope -1 and are guides to the eye.

In the type of queue we are considering, the variance in the number of packets in any node is given by [12]

$$\sigma_j = \frac{\sqrt{\mu_j}}{1-\mu_j} = \frac{\sqrt{\frac{\rho B_j}{S-1}}}{1-\frac{\rho B_j}{S-1}}. \tag{12}$$

Since the most connected nodes are those with the highest effective betweenness and its value determines the critical value of ρ, we expect that at these nodes

$$\sigma_{\text{m.c.n.}} \simeq \frac{1}{\rho_c - \rho}. \tag{13}$$

and this is precisely the behavior we infer from Fig. 1.

On the other hand, for the total number of packets, there a clear distinction between the two networks. In the centralized one, there are just a few nodes, three in our case, that carry most of the network load and hence the main contributions to the variance, making that the variance in the total number of packets follows the same law as the single most connected nodes. Contrary to this, for the distributed network there are much more nodes with a critical load and correlation are induced that may disrupt the single node behavior.

7 Conclusion

We have presented a single model that can handle simultaneously search and congestion in a general network. The dynamical properties of the network can be

related to purely topological properties, in terms of the connectivity matrix and on the search algorithm. Close to the critical point where congestion develops the system presents an interesting behavior since the variance of the total number of packets cannot be easily related to the variance of the number of packets in the single nodes.

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