Exact long-time behavior of a network of phase oscillators under random fields

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We present an exact solution for the order parameters that characterize the stationary behavior of a population of Kuramoto's phase oscillators under random external fields [Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki, Lecture Notes in Physics Vol. 39 (Springer, Berlin, 1975), p. 420]. From these results it is possible to generate the phase diagram of models with an arbitrary distribution of random frequencies and random fields.

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I. INTRODUCTION

The collective behavior of large assemblies of oscillators coupled through nonlinear interactions and under the effect of external fields has been understood from a qualitative point of view in multiple situations, but the development of a mathematical formalism that could provide exact expressions for the variables that describe the long-time properties of specific models is not an easy task. It is true that for systems with a small number of elements an important set of analytical studies has appeared in the last decades. Most of them aimed at finding chaos, synchronization, and other dynamical effects. Sometimes the calculation of the trajectories of pairs of units (with complex individual dynamics) gives information about the global temporal behavior. This strategy has been successful in the study of synchronization (entrainment) [1, 2] in heart pacemaker cells and other biological systems [3]. However, it is obvious that this approach is not practical for large populations and other ideas are necessary for a satisfactory description of them.

Among the systems that may present synchronization phenomena, models of phase oscillators are probably the most deeply analyzed. As a consequence, it is in this context where have appeared some of the most relevant contributions to the mathematical description of this cooperative effect [4, 6, 5]. These models are intrinsically simple because it is assumed that the interaction between oscillators is sufficiently weak to consider that all of them describe a trajectory along a global attracting limit cycle of constant amplitude. Therefore, to analyze their dynamical evolution it is only necessary to have information about the variations of the respective phases. Winfree [7] pointed out that such simplified models may account for patterns of collective synchronization similar to those observed in large groups of biological beings (swarms of fireflies, temporal activity of pacemaker cells, etc.). These ideas have also been applied to other fields of physics (Josephson junctions, charge density waves, neural networks, etc.).

The general interest has been basically focused on two kinds of systems. One is models of integrate-and-fire oscillators with all-to-all excitatory couplings. It has been shown that, under certain conditions, perfect synchrony between the elements of the population is achieved in the stationary state [5]. This cooperative phenomenon has also been studied when the units are subject to external noise [8]. More recently, Van Vreeswijk and Abbott proved that self-sustained firing is also possible even when the excitable units are not intrinsic oscillators [9]. The essential ingredient in all these models is the nonlinear character of the pulselike interaction between elements.

On the other hand, large assemblies of limit-cycle oscillators (active planar rotators) interacting with each other through couplings proportional to their difference of phases define a different type of model whose best known example is the so-called Kuramoto model (KM) [6]. According to the KM, the phase of each element of the population obeys the following Langevin equation:

$$\dot{\theta}_i = \omega_i + \gamma_i(t) + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i), \tag{1}$$

where K_{ij} is the coupling matrix, θ_i the phase of the ith oscillator, ω_i is the random intrinsic frequency [distributed as $g(\omega)$] of the ith oscillator, N the size of the population, and $\gamma_i(t)$ independent white noise random processes with zero mean and correlation

$$\langle \gamma_i(t)\gamma_j(t')\rangle = 2D\delta_{ij}\delta(t-t'), \qquad D \geq 0. \tag{2}$$

This model has been analyzed in a wide range of situations. Shinomoto *et al.* have shown through qualitative arguments that, with short-range interactions, a macroscopic degree of synchronization turns out to be impossible below $d = 3$ [10, 11]. This statement has been reinforced by Daido [12] who applied renormalization group techniques to find the dependence of the lower critical dimension d_c on the characteristic exponent of the distribution of frequencies. For the usual distributions $d_c = 2$. In lattices of higher dimensionality Strogatz and Mirollo [13] observed that phase locking develops in spongelike structures. This is different from what one expects for a mean field theory, suggesting that the upper critical dimension might be $d = \infty$.

For long-range interactions, a mean field formalism allows one to find rigorously (for $N \to \infty$) the effect that all the parameters appearing in (1) have on the cooperative

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behavior of the system. In the presence of ferromagnetic interactions ($K_{ij} = K/N$) and a unimodal distribution of intrinsic frequencies, it has been shown [14] that for a critical value of the ratio K/D a phase transition occurs from a state where all the oscillators run incoherently to another state where a certain degree of synchronization appears spontaneously. Other densities $g(\omega)$ may induce a more complex character to the transition. For instance, Bonilla *et al.* [15] have shown through a bifurcation analysis that a bimodal distribution introduces bistability, oscillatory behavior, and hysteretic phenomena.

The influence of random couplings has also been analyzed in recent studies. The main effect of randomness and frustration is the appearance of new phases (for instance, a glassy phase where global synchronization is zero but the state defined by the oscillators is correlated with the random disorder), reminiscent of magnetic systems such as spin glasses [16,17]. Other important contributions have appeared in the field of neural networks. By suggesting more complex prescriptions for the couplings, models of phase oscillators have reproduced through simulations the oscillatory properties of biological neurons and how an external stimulus can produce coherence in their temporal activity [18–20].

Finally, a special comment is deserved by the studies made by Strogatz *et al.* [21] on the effect of random pinning in charge density waves and by Sakaguchi [22] on the influence of periodic forces on the behavior of a population of planar rotators. The similarities and differences with our work will be discussed in this paper.

Our goal is to present a mathematical formalism that allows one to compute analytically the properties of a population of KM phase oscillators under external fields. The analysis is performed in the thermodynamic limit. We give exact expressions for the order parameters that describe the long-time behavior of the system. The structure of our procedure has been used before in [23] to find the phase diagram of a model without frequency differences. As we will show later, to introduce a zero mean distribution $g(\omega)$ does not change the relevant features of such a phase diagram. However, if $\langle w \rangle \neq 0$ new dynamical effects must be considered.

The paper is organized as follows. Section II is devoted to the definition of the model and of the relevant order parameters. In Sec. III we give a continuous description of the model in terms of a nonlinear Fokker-Planck equation for the one-oscillator probability density. A generating functional defined to find algebraic expressions for the order parameters is introduced in Sec. IV. A comparison between our analytical results and Brownian simulations is performed in Sec. V. Finally, we discuss the range of validity of our approach in relation to the features of $g(\omega)$ and the distribution of random fields.

II. THE MODEL

Motivated by the analogy between Kuramoto's model of active rotators and the planar XY model of spins studied previously in [23], we have investigated the properties of a system of phase oscillators with all-to-all ferromagnetic couplings in the presence of a random external field, distributed according to a certain probability density $f(h)$. The dynamical evolution of each element of the population is given by

$$\dot{\theta}_i = \omega_i + \gamma_i(t) + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i) - h_i \sin\theta_i, \tag{3}$$

where h_i denotes the field acting on the ith site, and $K_{ij} = K/N$. As a simplification, we have assumed a fixed direction for the whole set of h_i, although our approach can be generalized to more complex spatial distributions (for instance a random p-fold symmetry breaking field). For now, we assume that the field does not have an explicit dependence on time. The effect of periodical external forces will be discussed in Sec. VI.

To find the long-time properties of the model we have followed a mean field formalism whose basic lines have been sketched in previous works [15,16,23], although here there are some technical details that make the mathematical calculations more difficult. The usual description of the system is done in terms of the following order parameters:

$$re^{i\phi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \tag{4}$$

where r gives the degree of synchronization in the population, and ϕ is a mean phase. In terms of these quantities, Eq. (3) reads

$$\dot{\theta}_i = \omega_i + kr\sin(\phi - \theta_i) - h_i \sin\theta_i + \gamma_i(t), \tag{5}$$

which is the starting point of our approach. Notice that the cooperative behavior of the system is a competition between the distribution of frequencies, the coupling strength K, the amount of noise, and the external field. The coupling tends to put in phase all the elements of the population, whereas the dispersion of frequencies and the noise tend to destroy coherence. The random field breaks the symmetry by imposing a privileged orientation.

Strogatz *et al.* [21] have considered a similar situation to analyze the effect of random pinning in models of charge density wave transport. However, their study is performed in the absence of random frequencies, with zero noise, and the fields are of constant modulus. Moreover, they consider either static solutions $\dot{\theta}_i = 0, \ \forall\, i$ (a particular case of our analysis) and time dependent solutions with constant r [we observe states with $r(t)$].

III. FOKKER-PLANCK DESCRIPTION

The first step is to transform the discrete description given by Langevin equation (5) into a continuous description. In the thermodynamic limit it is possible to derive a nonlinear Fokker-Planck equation for the one-oscillator probability density $\rho(\theta, t, \omega, h)$. It gives the probability that an oscillator with frequency ω and under the action of h has a phase θ at an instant t.

The idea, illustrated in [24], consists of writing $\rho(\theta, t, \omega, h)$ in terms of the N-oscillator probability density ρ_N, the solution of the Fokker-Planck equation associated with (3), as the product of the N one-oscillator probability densities. This is possible assuming the propagation of molecular chaos (this means that the one-oscillator probability densities are not correlated) from the initial conditions. Then, by writing the path integral representation of ρ_N in the resulting expression and performing the integrals through the steepest descent method in the limit of $N \to \infty$, we have found that the final expression for $\rho(\theta, t, \omega, h)$ must satisfy

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}[\mathcal{V}\rho] - D\frac{\partial^2 \rho}{\partial \theta^2} = 0, \tag{6}$$

where $\mathcal{V}$ is

$$\mathcal{V} = [\omega + Kr\sin(\phi - \theta) - h\sin\theta]. \tag{7}$$

Our goal is to obtain the stationary behavior of the system described by (6) and (7). Stationary means that $\rho(\theta, t, \omega, h)$ does not have an explicit dependence on time or does behave periodically. The first case is observed when the distribution of frequencies is unimodal and symmetric, conditions that are satisfied by the most usual ones (uniform, Gaussian, etc.). Temporal periodicity is observed in other situations discussed in Sec. VI. Our approach deals with the first situation, when the one-oscillator probability density satisfies $(\partial\rho/\partial t) = 0$. The solution of (6) with the previous condition is straightforward. Normalization and 2π periodicity in θ lead to [15, 16]

$$\rho(\theta, \omega, h) = \frac{F(\theta)\int_0^{2\pi} d\eta H(\theta, \eta)}{\mathcal{Z}}, \tag{8}$$

where $F(\theta)$, $H(\theta, \eta)$, and $\mathcal{Z}$ are defined as

$$F(\theta) = \exp\left[\frac{K}{D}r\cos(\phi - \theta) - \frac{h}{D}\cos\theta\right], \tag{9}$$

$$H(\theta, \eta) = \exp\left[-\frac{\omega\eta}{D} - \frac{K}{D}r\cos(\phi - \theta - \eta) + \frac{h}{D}\cos(\theta + \eta)\right], \tag{10}$$

and

$$\mathcal{Z} = \int_0^{2\pi} d\theta F(\theta)\int_0^{2\pi} H(\theta, \eta)d\eta. \tag{11}$$

The order parameters r and ϕ in terms of ρ and the distributions $g(\omega)$ and $f(h)$ are given by

$$re^{i\phi} = \int \ldots \int e^{i\theta}\rho(\theta, \omega, h)d\theta g(\omega)d\omega f(h)dh. \tag{12}$$

These self-consistent equations provide all the relevant information of the system. However, they contain an explicit dependence on θ that can be eliminated by direct integration. The most elegant way to perform the integral over θ is explained in detail in the next section.

IV. GENERATING FUNCTIONAL

The method proposed in this section takes advantage of the normalized structure of the probability density. We have defined a functional whose derivatives (more exactly the derivatives of its logarithm) allow one to compute the order parameters in a straightforward manner. The form of this generating functional is

$$\mathcal{Z}[\sigma, \xi] = \int_0^{2\pi} d\theta F(\sigma, \theta, \xi, \psi)\int_0^{2\pi} H(\theta, \eta)d\eta, \tag{13}$$

where

$$F(\sigma, \theta, \xi, \psi) = \exp[\sigma\cos(\psi - \theta) - \xi\cos\theta] \tag{14}$$

and H is given by (10). σ, ψ, and ξ are the three parameters with respect to which we perform the derivatives. According to these definitions r and ϕ are calculated from $\ln\mathcal{Z}$ as

$$r = \left\langle\!\!\left\langle \frac{\partial}{\partial\sigma}\ln\mathcal{Z}\,\Big|_{\sigma=\frac{kr}{D},\xi=\frac{h}{d},\psi=\phi} \right\rangle\!\!\right\rangle, \tag{15}$$

$$\phi = \arctan\left(\frac{\langle\!\langle r\sin\phi + \cos\phi\frac{\partial}{\partial\psi}\ln\mathcal{Z}\,|_{\sigma=\frac{Kr}{D},\xi=\frac{h}{d},\psi=\phi}\rangle\!\rangle}{\langle\!\langle -\frac{\partial}{\partial\xi}\ln\mathcal{Z}\,|_{\sigma=\frac{kr}{D},\xi=\frac{h}{d},\psi=\phi}\rangle\!\rangle}\right) \tag{16}$$

where $\langle\!\langle\cdots\rangle\!\rangle$ is an average over the distributions $g(\omega)$ and $f(h)$. This approach has several advantages since it allows one to simplify notably the numerical resolution of the resultant expressions and to perform an analysis of them.

The basic details of the calculation of $\mathcal{Z}$ have been discussed previously in [23] for a population of XY spins. Here, the mathematics is more involved since the existence of intrinsic frequencies introduces an asymmetry between F and H. By using only symmetry properties of the spherical modified Bessel functions $I_m(x)$ and some simple algebra, the set of terms generated from direct integration of the functional (see the Appendix) can be written in a compact form as

$$\mathcal{Z} = (1 - e^{\frac{2\pi\omega}{D}})\left[\frac{-\pi}{2}\sum_{m=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(-1)^{(m+p)} \times I_m\left(\frac{kr}{D}\right)I_p(\xi)I_l(\sigma)\Delta^{(l,m,p)}\right], \tag{17}$$

where

$$\Delta^{(l,m,p)} = I_{l+p-m}\left(\frac{h}{D}\right)\frac{-\frac{w}{D}\cos(l\psi - m\phi) + (l+p)\sin(l\psi - m\phi)}{(\frac{w}{D})^2 + (l+p)^2} + I_{l-p-m}\left(\frac{h}{D}\right)\frac{-\frac{w}{D}\cos(l\psi - m\phi) + (l-p)\sin(l\psi - m\phi)}{(\frac{w}{D})^2 + (l-p)^2}$$

$$+I_{l+p+m}\left(\frac{h}{D}\right)\frac{-\frac{w}{D}\cos(l\psi+m\phi)+(l+p)\sin(l\psi+m\phi)}{(\frac{w}{D})^2+(l+p)^2}$$
$$+I_{l-p+m}\left(\frac{h}{D}\right)\frac{-\frac{w}{D}\cos(l\psi+m\phi)+(l-p)\sin(l\psi+m\phi)}{(\frac{w}{D})^2+(l-p)^2}. \tag{18}$$

Now, it is easy to compute r and ϕ from formulas (15) and (16). The final expressions are

$$r=\int\int\frac{\sum\sum\sum(-1)^{(m+p)}I_m(\frac{kr}{D})I_p(\frac{h}{D})I_{l+1}(\frac{kr}{D})\Delta^{(l,m,p)}}{\sum\sum\sum(-1)^{(m+p)}I_m(\frac{kr}{D})I_p(\frac{h}{D})I_l(\frac{kr}{D})\Delta^{(l,m,p)}}g(\omega)f(h)d\omega dh \tag{19}$$

and

$$\phi=\arctan\left(\frac{r\sin\phi+\mathcal{A}\cos\phi}{\mathcal{B}}\right), \tag{20}$$

where

$$\mathcal{A}=\int\int\frac{\sum\sum\sum(-1)^{(m+p)}I_m(\frac{kr}{D})I_p(\frac{h}{D})I_l(\frac{kr}{D})(\partial\Delta^{(l,m,p)}/\partial\psi)}{\sum\sum\sum(-1)^{(m+p)}I_m(\frac{kr}{D})I_p(\frac{h}{D})I_l(\frac{kr}{D})\Delta^{(l,m,p)}}g(\omega)f(h)d\omega dh \tag{21}$$

and

$$\mathcal{B}=\int\int\frac{\sum\sum\sum(-1)^{(m+p)}I_m(\frac{kr}{D})I_{p+1}(\frac{h}{D})I_l(\frac{kr}{D})\Delta^{(l,m,p)}}{\sum\sum\sum(-1)^{(m+p)}I_m(\frac{kr}{D})I_p(\frac{h}{D})I_l(\frac{kr}{D})\Delta^{(l,m,p)}}g(\omega)f(h)d\omega dh, \tag{22}$$

where the series run over the three indices m, l, and p. These equations provide all the information necessary to reproduce the stationary behavior of the system. From them we can compute the phase diagram of any system made of Kuramoto phase oscillators characterized by an arbitrary distribution of frequencies and external random fields, provided the order parameters r and ϕ do not have an explicit dependence on time. Both equations can be solved self-consistently by standard techniques such as the Newton-Raphson method.

Looking at expressions (19)–(22) one may think that the solution of the problem is as complex as before (without the integration over θ) and that it is unapproachable from a numerical point of view. However, this is only a superficial analysis, far away from reality. These series present an excellent behavior due to their convergence properties that make them easy to deal with. For instance, by considering less than ten terms one can get results with an accuracy of order 10^{-4}. This fact simplifies the situation notably.

V. COMPARISON WITH SIMULATIONS

In order to check the accuracy of our approach, we have compared the results achieved from the numerical resolution of Eqs. (19) and (20) with Brownian simulations. As an example, we have considered two different situations. In both cases the distribution of frequencies is uniform and even in the range [−0.5,0.5] while the fields are also uniformly distributed with the same variance but centered at 0 and at 0.25 in the first and the second cases, respectively. The ratio K/D has been controlled by fixing the amount of noise to $D = 0.2$ and moving K.

To integrate Eqs. (19) and (20) we have taken the first ten terms of the series. Brownian simulations have been performed by solving the stochastic equation (3) with a Euler method with a time step $\delta t = 0.05$. We have considered a network of 20 000 oscillators, large enough

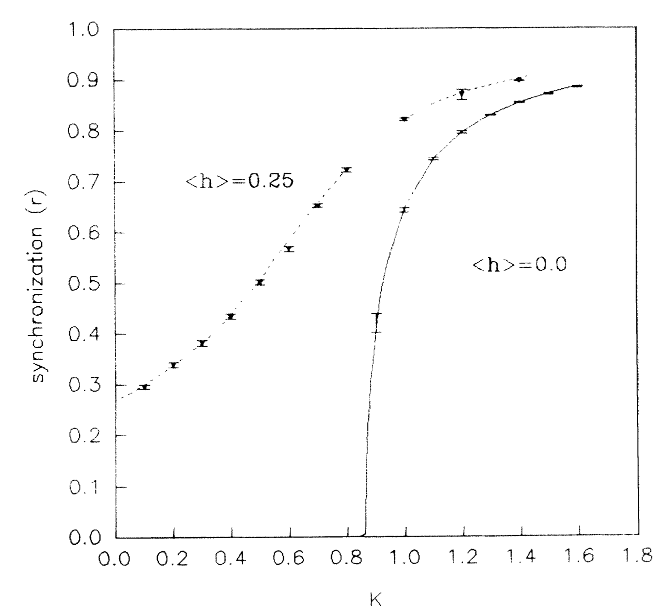

FIG. 1. Degree of synchronization versus intensity of couplings for a population of KM oscillators under the action of a random field distributed uniformly in [−0.25,0.75] (dashed line) and [−0.5,0.5] (continuous line), and with intrinsic frequencies and noise described in the text. Lines represent results coming from the numerical solution of Eqs. (19) and (20), while symbols are obtained from simulations. Each symbol has been computed as an average over 1000 time steps in the stationary state.

to neglect finite size effects. The results are plotted in Fig. 1.

This figure deserves several comments. First, we observe an excellent agreement between the theoretical expressions and the simulations, which shows that our results are exact in the thermodynamic limit. Additionally, it is interesting to remark that introducing a one-humped distribution of frequencies does not modify the qualitative behavior of the static XY model previously studied in [23]. With a centered (at the origin) distribution of fields a phase transition between an incoherent state with $r=0$ and another with a macroscopic degree of synchronization (magnetization) still takes place and the only difference with respect to the static case is the situation of the transition point, displaced toward a higher value of K. This is evident since larger values of K are necessary to counteract the effect of rotation caused by the distribution of frequencies.

When the distribution of fields is not centered at the origin there is an effective force which makes $r \neq 0$ for any value of the ratio K/D (in fact for small K/D and for $\langle h \rangle \ll 1$, then r is proportional to $\langle h \rangle$). The effect of the variance of $g(\omega)$ is to reduce the coherence among members of the population (i.e., to reduce r), although it does not change the qualitative behavior observed in Fig. 1. In conclusion, no phase transition occurs, just as in the static case.

VI. DISCUSSION

As we have shown in the previous section, Eqs. (19) and (20) allow one to generate the phase diagram of any system of phase oscillators described by the Langevin equation (3) under the conditions of stationarity. In this section we want to discuss whether such conditions are fulfilled for any distribution of random fields and intrinsic frequencies, since the answer will give us information about the range of validity of our approach.

We have found two basic mechanisms that could lead to time dependent order parameters: either by introducing an external time-periodic force or by considering the existence of a complex distribution of frequencies. However, the first one is equivalent to assuming a fixed external field and a redefinition of the intrinsic frequencies. To prove this statement let us consider the following model proposed by Sakaguchi [22]:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) - h\sin(\theta_i - \omega_f t), \quad (23)$$

where ω_f denotes the external driving. By introducing a new phase $\psi_i = (\theta_i - \omega_f t)$, we observe that the previous description is transformed to a static one equivalent to (3), where the explicit dependence on time of the external force is absorbed in the features of a new distribution of frequencies. Therefore, the discussion reduces to considering a suitable $g(\omega)$.

We have observed that if the transformation previously mentioned generates a one-humped, nonincreasing distribution centered at a certain $\omega_c \neq 0$ the system presents a periodic behavior whose description is complex because the period of this oscillation is not directly related to $1/\omega_c$. To analyze this situation it is necessary to solve the complete time dependent Fokker-Planck equation (6) or to apply a different technique that could enable us to find $r(t)$ and $\phi(t)$. However, what we want to stress in this section is that with our approach it is possible to reproduce the oscillating behavior of r through an explicit knowledge of the temporal variation of the mean phase.

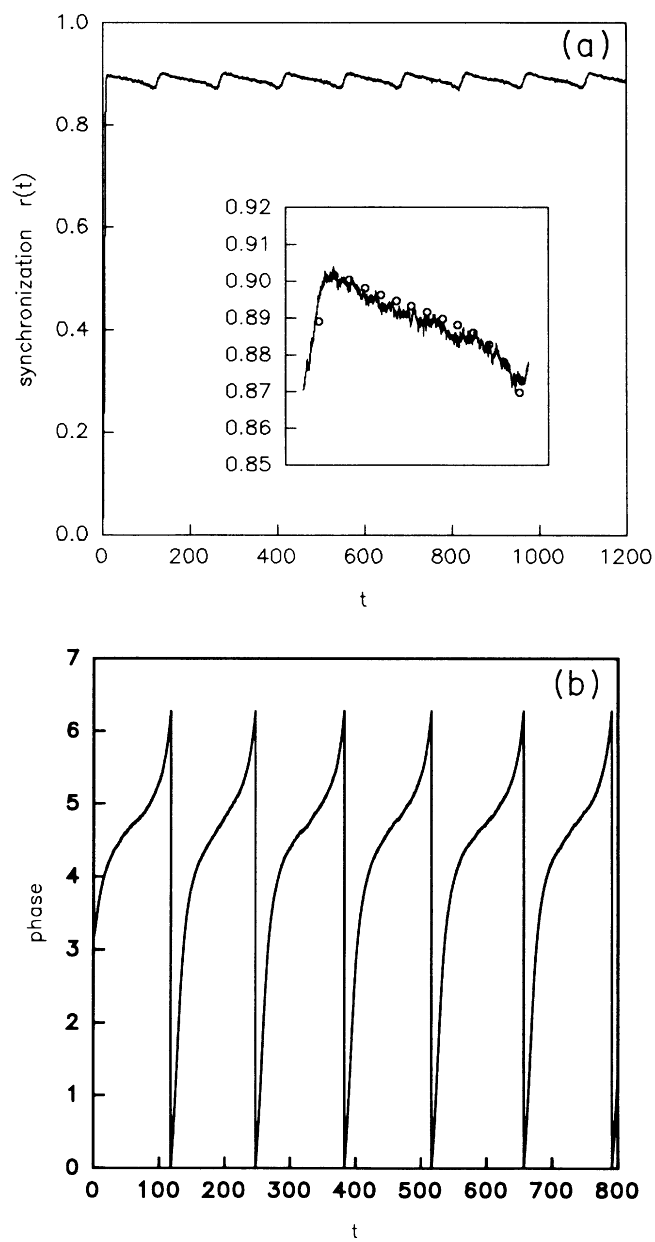

FIG. 2. (a) Temporal evolution of the order parameter r for a population of phase oscillators with the distribution of intrinsic frequencies and fields mentioned in the text. Other parameters are K=1.5, D=0.2, and N=20 000. In the inset we have represented, for a period, the results obtained by embedding the phases of (b) in Eq. (19) (circles) versus the continuous line from the Brownian simulation. (b) Temporal evolution of the mean phase for the system described in (a).

In Figs. 2(a) and 2(b) we have shown the evolution of r and ϕ for a uniform distribution of frequencies between [−0.4,0.6] and a fixed external field $h_i = 0.1$ $\forall i$. Time is expressed in seconds (a second for us corresponds to 200 iterations). We observe the nontrivial periodic character of both parameters. However, if for a given t we pick the corresponding $\phi(t)$ [in this example from Fig. 2(b)] and we put this numerical result into Eq. (19) we reproduce completely the behavior of $r(t)$. This means that the functional relationship between both order parameters given by Eq. (19) still holds and that all the time dependence of $r(t)$ comes through the mean phase $\phi(t)$. Then the problem reduces to finding a dynamical equation for $\phi(t)$. We believe that this situation appears due to the simple structure of $g(\omega)$ and that more complex distributions of random frequencies (for instance, a bimodal) could introduce dynamical effects not properly reproduced by expressions (19) and (20). Nevertheless, a linear stability analysis of them may show the location of bifurcation points which allow one to identify the transition between phases with stationary-time dependent order parameters. This study deserves special attention.

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APPENDIX

In this section, we want to describe some technical details which allow us to derive Eqs. (19) and (20) from the definition of the generating functional (17). The first step is to write the exponentials (14) and (10) in terms of series of spherical modified Bessel functions [25];

$$e^{x\cos(\xi)} = I_0(x) + 2\sum_{n=1}^{\infty}(-1)^n \cos(n\xi) I_n(x), \tag{A1}$$

and perform integration (13) directly. This integral produces the following set of terms:

$$\frac{2\pi D}{\omega} I_0\left(\frac{kr}{D}\right) I_0\left(\frac{h}{D}\right) I_0(\xi) I_0(\sigma), \tag{A2}$$

$$\frac{4\pi D}{\omega}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m I_m\left(\frac{kr}{D}\right) I_n\left(\frac{h}{D}\right) I_0(\xi) I_0(\sigma)\cos(m\phi)\delta_{n,m}, \tag{A3}$$

$$\frac{4\pi D}{\omega}\sum_{p=1}^{\infty}\sum_{l=1}^{\infty}(-1)^p I_0\left(\frac{kr}{D}\right) I_0\left(\frac{h}{D}\right) I_p(\xi) I_l(\sigma)\cos(l\psi)\delta_{l,p}, \tag{A4}$$

$$4\pi\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}(-1)^p I_0\left(\frac{kr}{D}\right) I_n\left(\frac{h}{D}\right) I_p(\xi) I_0(\sigma)\frac{\frac{w}{D}}{\frac{w}{D}^2+n^2}\delta_{n,p}, \tag{A5}$$

$$-4\pi\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}(-1)^{m+p} I_m\left(\frac{kr}{D}\right) I_0\left(\frac{h}{D}\right) I_p(\xi) I_0(\sigma)\frac{-\frac{w}{D}\cos(m\phi)-m\sin(m\phi)}{\frac{w}{D}^2+m^2}\delta_{m,p}, \tag{A6}$$

$$-4\pi\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} I_0\left(\frac{kr}{D}\right) I_n\left(\frac{h}{D}\right) I_0(\xi) I_l(\sigma)\frac{-\frac{w}{D}\cos(l\psi)-n\sin(l\psi)}{\frac{w}{D}^2+n^2}\delta_{n,l}, \tag{A7}$$

$$4\pi\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}(-1)^m I_m\left(\frac{kr}{D}\right) I_0\left(\frac{h}{D}\right) I_0(\xi) I_l(\sigma)\frac{\frac{w}{D}}{\frac{w}{D}^2+m^2}\delta_{l,m}, \tag{A8}$$

$$\begin{aligned}&-4\pi\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{m+p} I_m\left(\frac{kr}{D}\right) I_0\left(\frac{h}{D}\right) I_p(\xi) I_l(\sigma)\\ &\times\left(\frac{-\frac{w}{D}\cos(p\phi)-m\sin(p\phi)}{\frac{w}{D}^2+m^2}\delta_{m,l+p}+\frac{-\frac{w}{D}\cos(p\phi)+m\sin(p\phi)}{\frac{w}{D}^2+m^2}\delta_{m,l-p}\right),\end{aligned} \tag{A9}$$

$$- 4\pi \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{p} I_0\left(\frac{kr}{D}\right) I_n\left(\frac{h}{D}\right) I_p(\xi) I_l(\sigma)$$

$$\times \left(\frac{-\frac{w}{D}\cos(l\psi) - n\sin(l\psi)}{\frac{w}{D}^2 + n^2}\delta_{n,l+p} + \frac{-\frac{w}{D}\cos(l\psi) + n\sin(l\psi)}{\frac{w}{D}^2 + n^2}\delta_{n,l-p}\right), \quad \text{(A10)}$$

$$- 4\pi \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}(-1)^{p+m} I_m\left(\frac{kr}{D}\right) I_n\left(\frac{h}{D}\right) I_p(\xi) I_0(\sigma)$$

$$\times \left(\frac{-\frac{w}{D}\cos(m\phi) - (n+m)\sin(m\phi)}{\frac{w}{D}^2 + (n+m)^2}\delta_{n+m,p} + \frac{-\frac{w}{D}\cos(m\phi) + (n+m)\sin(m\phi)}{\frac{w}{D}^2 + (n+m)^2}\delta_{n-m,p}\right), \quad \text{(A11)}$$

$$- 4\pi \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{m} I_m\left(\frac{kr}{D}\right) I_n\left(\frac{h}{D}\right) I_0(\xi) I_l(\sigma)$$

$$\times \left(\frac{-\frac{w}{D}\cos(n\phi) + (n+m)\sin(n\phi)}{\frac{w}{D}^2 + (n+m)^2}\delta_{n+m,l} + \frac{-\frac{w}{D}\cos(n\phi) + (n-m)\sin(n\phi)}{\frac{w}{D}^2 + (n-m)^2}\delta_{n-m,l}\right), \quad \text{(A12)}$$

$$- 4\pi \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{m+p} I_m\left(\frac{kr}{D}\right) I_n\left(\frac{h}{D}\right) I_p(\xi) I_l(\sigma)$$

$$\times \left(\frac{-\frac{w}{D}\cos(l\psi - m\phi) + (n+m)\sin(l\psi - m\phi)}{\frac{w}{D}^2 + (n+m)^2}\delta_{n+m,l+p} + \frac{-\frac{w}{D}\cos(l\psi - m\phi) + (n+m)\sin(l\psi - m\phi)}{\frac{w}{D}^2 + (n+m)^2}\delta_{n+m,l-p}\right.$$

$$\left. + \frac{-\frac{w}{D}\cos(m\phi + l\psi) + (n-m)\sin(m\phi + l\psi)}{\frac{w}{D}^2 + (n-m)^2}\delta_{n-m,l+p} + \frac{-\frac{w}{D}\cos(m\phi + l\psi) + (n-m)\sin(m\phi + l\psi)}{\frac{w}{D}^2 + (n-m)^2}\delta_{n-m,l-p}\right). \quad \text{(A13)}$$

Now, by using the contraction of the Kronecker delta $\delta_{n,m}$ and the symmetry properties of the modified Bessel functions with integer index $[I_n(x) = I_{-n}(x)]$, we can extend $\sum_1^\infty$ to $\sum_{-\infty}^\infty$ taking into account the corresponding zero terms. In this way we can reduce the whole set of terms (A2)–(A13) to a compact expression for $\mathcal{Z}$ given by (17). In more detail, expression (A2) corresponds to taking $m = p = l = 0$ simultaneously in (17). Expressions (A3), (A5), and (A7) come out by taking (i) $l = p = 0$, $m \neq 0$, (ii) $l = m = 0$, $p \neq 0$, and (iii) $m = p = 0$, $l \neq 0$, respectively, in (17). The rest of the terms can be obtained by applying the same procedure. In particular, the last one corresponds to the nonzero index contributions.

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