Synchronization in a Lattice Model of Pulse-Coupled Oscillators

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We analyze the collective behavior of a lattice model of pulse-coupled oscillators. By means of computer simulations we find the relation between the intrinsic dynamics of each member of the population and their mutual interactions that ensures, in a general context, the existence of a fully synchronized regime. This condition turns out to be the same as that obtained for the globally coupled population. When the condition is not completely satisfied we find different spatial structures. This also gives some hints about self-organized criticality.

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The collective behavior of large assemblies of pulse-coupled oscillators has been investigated quite often in the last years. Many physical and biological systems can be described in terms of populations of units that evolve in time according to a certain intrinsic dynamics and interact when they reach a threshold value [1]. Although it was known long ago that the members of these systems tend to have a synchronous temporal activity, a rigorous treatment of the problem has been considered only in the last decade [2–4]. Up to now, the most important efforts have been focused on systems with long-range interactions, because in this case analytical results can be derived by applying a mean-field formalism. Relevant is the work by Mirollo and Strogatz (MS) [4] who discovered under which conditions mutual synchronization emerges as the stationary configuration of the population. Later the study was generalized to other situations [5–10].

When the oscillators form a finite-dimensional lattice, where only short-range interactions are allowed, the spectrum of behaviors is more complex. For instance, some recent papers have related lattice models of pulse-coupled oscillators to models displaying self-organized criticality (SOC) [11–15], systems which self-organize, due to its own dynamics, into a critical state with no characteristic time or length scales [16]. Of particular interest for us is the model proposed by Feder and Feder (FF) [17] to study stick-slip processes in earthquake dynamics. In such a model each cell of a 2D square lattice is described physically in terms of a state variable $E(t)$ (hereafter called energy) that evolves linearly with time. Once the energy of a cell reaches a threshold value ($E_{i,j} \geq E_c = 1$) it becomes critical and "fires" transferring energy to its nearest neighboring cells according to the following rules:

$$E_{nn} \rightarrow E_{nn} + \varepsilon\,, \qquad E_{i,j} \rightarrow 0\,, \tag{1}$$

where ε is the strength of the coupling In turn, some of these neighboring cells may become critical, generating an avalanche that propagates through the lattice. When one avalanche is triggered, the intrinsic dynamics is stopped, and only when it is over does the driving act again. In this way, there is a clear separation of two time scales. By identifying the state variable E with a voltagelike magnitude, one can establish an analogy between the FF model and some models of integrate-and-fire oscillators.

Without any other ingredient the FF model displays, in the stationary state, and for open boundary conditions, relaxation oscillations (RO) which give rise to spatial structures formed by large assemblies of units, all with the same phase. In this sense we could talk about a macroscopic (local) degree of synchronization. However, when a dynamical noise is added to the system new collective behaviors appear, since the synchronized state is no longer an attractor of the dynamics. For instance, for $\varepsilon = 0.25$ [11], the distribution of avalanche sizes follows a power-law decay characteristic of SOC.

In the short-range models described above the energy of each oscillator varies with a constant driving rate $f(E) \equiv dE/dt = C$. The analysis has been extended to linear driving rates [12] finding new conditions to observe RO even in the presence of noise. It would be interesting to consider a more general context, where $f'(E)$ can take an arbitrary shape with the only restriction being $f(E) > 0$. There is another underlying hypothesis in (1) that restricts the range of interest of these models: the strength of the coupling ε is constant. In many studies of biological pacemakers it is assumed that the response of a cell due to an external stimulus is a function of the phase, $\Delta(\phi)$, which depends on the current state of the oscillators [18,19]. The phase shift induces an energy shift $\varepsilon(E)$ [hereafter called the energy response curve (ERC)] that, in general, is a nonconstant function of E. Our goal is to investigate a wide variety of models with arbitrary ERC and driving rate $f(E)$ beyond those considered in previous works. Although in such a general situation a broad range of different regimes can be observed, we will focus our study on the conditions required for the system to develop a fully synchronous stationary state, emphasizing the relevance of the response of a given oscillator at its reset point, $\varepsilon(E = 0)$. The analysis also provides information about the possible nature of SOC. Our interest mainly concerns coupled map lattice models, but to start the discussion a mean-field model is introduced to clarify several dynamic aspects.

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Let us consider a system of globally coupled oscillators which interact through a given $\overline{\varepsilon}(E)$. The dynamics of each unit is given by

$$\frac{dE_i}{dt} = f(E_i) + \overline{\varepsilon}(E_i)\delta(t - t_j) \qquad (2)$$

plus the reset condition for $E_i \geq 1$. Here t_j denotes the time at which unit or group j fires. In this description both time scales (driving and coupling) appear in the same equation. We also could consider another equivalent description where both time scales are separated such as in (1), but this fact implies to substitute $\overline{\varepsilon}(E)$ by $\varepsilon(E)$ related through

$$\int_E^{E+\varepsilon(E)} \frac{dE'}{\overline{\varepsilon}(E')} = \int_{t_j^-}^{t_j^+} \delta(t - t_j)\,dt = 1\,. \qquad (3)$$

To go from $\overline{\varepsilon}(E)$ to $\varepsilon(E)$ is trivial, but the inverse implies to deal with an integral equation that, in general can only be solved numerically. Now, it is simple to derive sufficient conditions to ensure perfect entrainment between both oscillators. The method consists of applying the following transformation, used in different contexts by several authors [8,20]:

$$y = \varepsilon_0 \int_0^E \frac{dE}{\overline{\varepsilon}(E)}\,, \qquad (4)$$

where ε_0 is defined to ensure that $y(1) = 1$. By substituting into (2) we get

$$\frac{dy_i}{dt} = \varepsilon_0 \frac{f(E_i)}{\overline{\varepsilon}(E_i)} + \varepsilon_0 \delta(t - t_j)\,. \qquad (5)$$

Now the evolution of the system is described in terms of a new variable y for which the coupling is constant. A case of particular interest is that of a zero advance at the reset point $[\varepsilon(E = 0) = 0]$. In such a case, this transformation is well defined if the energy transferred $\forall y \neq 0$ is constant (ε_0) except for $y = 0$, which is exactly zero. This condition plays the role of a refractory time, which provokes that the units that have fired have zero phase at the end of the interaction process.

A perfect analogy can be established with the problem studied in MS [4]. They showed that for a constant positive coupling, no matter how small, the synchronized state is an absorbing state of the dynamics if the driving rate is positive and its first derivative negative. By applying theses conditions to the new function

$$g(y) = \varepsilon_0 \frac{f(E)}{\overline{\varepsilon}(E)}\,, \qquad (6)$$

we find the following relation between the driving rate and $\overline{\varepsilon}(E)$ that ensures the synchronization of the population:

$$\frac{f(E)}{\overline{\varepsilon}(E)} > 0 \quad \text{and} \quad \frac{f'(E)}{f(E)} < \frac{\overline{\varepsilon}'(E)}{\overline{\varepsilon}(E)} \quad \forall E\,. \qquad (7)$$

This means that given the features of the driving rate it is always possible to find the oscillators firing in unison provided one chooses the suitable ERC.

The interesting point is to know whether the mathematical result we have derived above can be extended to networks with short-range interactions. We have considered FF coupling (1) with $\varepsilon(E)$ and a nonconstant driving rate with dynamical noise [21]. It is important to realize that in contrast with mean-field models, where synchronization emerges in a process where clusters of oscillators of increasing size merge with each other (absorption process) and never break up, in a coupled map lattice model big assemblies of oscillators with the same phase (which eventually may break up) are generated through large RO that sweep the whole lattice (avalanches of the size of the system). Then for constant, instantaneous couplings, and due to the different nature of both mechanisms, the conditions required to find synchronization or RO are different [12]. However, the situation may change if one considers a nonconstant ERC. To analyze this new situation it is convenient to apply the same arguments discussed in [12], but now for a state-dependent coupling. For a square lattice with open boundary conditions, it is not difficult to show that a necessary condition to observe RO of the size of the system is

$$E[1 - \phi(4\varepsilon(0))] + 2\varepsilon[1 - \phi(4\varepsilon(0))] \geq 1\,, \qquad (8)$$

where the energy and the phase ϕ are related through the following expression:

$$\phi(E) = \int_0^E \frac{dE}{f(E)}\,. \qquad (9)$$

First of all, we observe the relevance of the response of a cell at the reset value. In general, a nonzero advance at this point $[\varepsilon(0) \neq 0)$ gives rise to a spatial distribution of phases after an avalanche; it is only under these conditions that SOC can be obtained [11]. On the other hand, it is clear that if $\varepsilon(0) = 0$ the inequality (8) is always satisfied for any driving rate and ERC. This means that the appearance of RO involving all the sites implies a perfect synchrony between all the elements of the lattice, boundaries included. However, (8) does not ensure the existence of those RO. It is a necessary but not a sufficient condition for the system to achieve a complete synchronization. We will check by simulations that condition (7) is, as in the long-range case, sufficient to produce synchrony.

Before giving evidence of this fact, we want to mention some conclusions that can be extracted from this result. Although, in general, one cannot map straightforwardly results from mean-field theories to coupled map lattices models, for these particular systems and as long as synchronization in concerned, conditions (7) which are strictly derived in a mean-field frame can be applied to short-range systems. This assumption is also in agreement with a conjecture proposed in [4] for the special case of $f'(E) < 0$ and $\overline{\varepsilon}'(E) = 0$.

Several models have been considered in our simulations, finding complete agreement with the theory. In particular, we have performed simulations for Peskin's model [22], where the driving rate is given by

$$f(E) = \gamma(K - E)\,, \qquad (10)$$

where γ gives the slope of the driving rate and $K = (1 - e^{-\gamma})^{-1}$. Starting from a random distribution of phases and for $\gamma > 0$ $[f'(E) < 0]$ we have observed that the population always synchronizes when $\overline{\varepsilon}(E)$ is a positive function of E with $\overline{\varepsilon}'(E) \geq 0$, provided that $\varepsilon(0) = 0$. However, for $\gamma = 0$ a monotonously increasing $\overline{\varepsilon}(E)$ is needed.

To complete this idea we have studied the time required for the population to synchronize, starting with random phases between 0 and 1, for an ERC given by

$$\overline{\varepsilon}(E) = \varepsilon_0 \gamma'(K' - E), \tag{11}$$

where $K' = (1 - e^{-\gamma'})^{-1}$ in a 32×32 lattice. We have plotted our results in Fig. 1 for open boundary conditions. There we can see how this time and the number of avalanches diverge as the difference between γ and γ' approaches zero. We have also checked a driving rate and an ERC both given by power laws, and the results are very similar; the time required to synchronize the system grows as $(\alpha - \alpha')^{-1}$, where α and α' are the exponents of the driving rate and of the ERC, respectively. Finally, we have simulated a driving rate of type (10) with a power-law ERC. For an exponent $\alpha' \geq 1$ the system synchronizes for any value of γ; in particular, the time needed to obtain the fully synchronized state grows exponentially as $e^{-\gamma/2}$ when $\alpha' = 1$. On the other hand, for $\alpha' < 1$ there is only a range of values of γ for which we get synchronization.

All these results are in agreement with our prediction (7). At this point it is important to remark that when the driving and the ERC have the same functional dependence

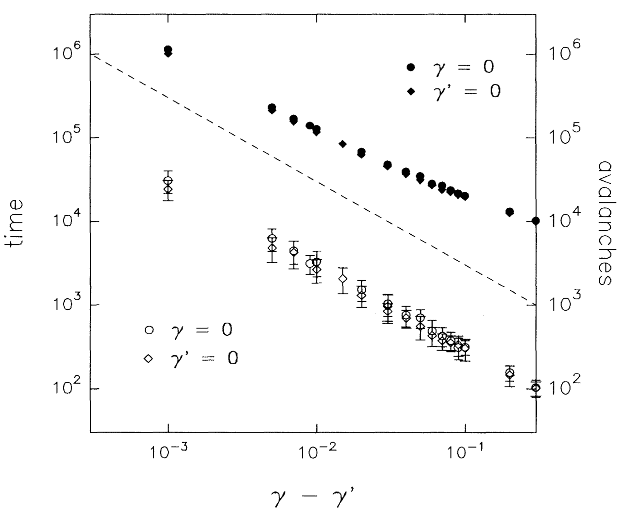

FIG. 1. Number of avalanches (filled) and time in period units (hollow) that a 32×32 lattice needs to get a complete synchronization as a function of $\gamma - \gamma'$ in a log-log scale. The symbols are averages over 25 realizations and the error bars (which for the avalanches are of the size of the symbol) correspond to the standard deviation. The straight line shows the $(\gamma - \gamma')^{-1}$ behavior.

on E the inequality can be satisfied for all E and the divergence of the time appears when both curves f'/f and $\overline{\varepsilon}'/\overline{\varepsilon}$ approach each other. However, when the dependence is different both curves can cross, and this leads to a more complex behavior. Finally, the case of a power-law ERC with $\alpha' \geq 1$ is very important because in this case the transformation (4) cannot be performed and therefore no mapping with the MS result can be done; nevertheless, (7) still represents the sufficient conditions to get a fully synchronized regime. Lattices with periodic boundary conditions have also been checked, and the conclusions are the same as for open ones; furthermore, the time required to synchronize scales in the same way.

We have also investigated the behavior of the model when one does not expect synchronization for a constant ERC, provided that $\varepsilon(E = 0) = 0$. First of all, let us recall the behavior of two coupled oscillators [4]. The phase of the one oscillator when the other one arrives to the threshold transforms according to

$$\phi_0 \to 1 - \phi[E(\phi_0) + \varepsilon] \tag{12}$$

provided that $E(\phi_0) + \varepsilon < 1$, otherwise the oscillators synchronize. This transformation has always at least one fixed point ϕ_0^* which leads to different behaviors depending on the slope of $f(E)$. Thus, for $f'(E) < 0$ $(\forall E)$, ϕ_0^* is unique and unstable, and the two oscillators will always synchronize for any positive value of ε [4]. However, for $f'(E) > 0$ $(\forall E)$ the stability of the fixed point changes, and ϕ_0^* becomes an attractor, which means that the oscillators can either be phase locked, when $\varepsilon < 1 - E(\phi_0)$, or synchronized, otherwise. For Peskin's model (10) ϕ_0^* corresponds to

$$\phi_0^* = 1/2 - (1/\gamma)\sinh^{-1}[\varepsilon \sinh(\gamma/2)]. \tag{13}$$

This result enables us to perform a qualitative analysis of the lattice model with nearest-neighbor interactions. Starting from a random distribution of phases, and only for periodic boundary conditions, the oscillators will tend to be locally synchronized or phase locked, depending on their phases and the parameters' values. This makes us suspect the existence of well-defined spatiotemporal structures of phase-locked oscillators in the stationary state, of the same kind as those described in [23], for which simple return maps can be written. The most simple of these configurations one can imagine is a chessboard lattice where "black" sites have the same value of the phase (ϕ_0) when all the "white" sites arrive to the threshold. Once the white ones have fired, the black ones are driven up to the threshold value and one gets the same structure as before with a phase ϕ_0 that transforms according to (12) replacing ε by 4ε. This means that there exists a fixed point for this structure, whose value is the same as that obtained from (13) replacing ε by 4ε, that is, an attractor for the dynamics of the lattice. It is important to remark that this structure is characterized by the fact that a given oscillator is phase locked with its four neighbors. Other simple structures arise when an oscillator can be synchronized with one neighbor and phase locked with the remaining ones (all these neighbors with the same phase), and so on, as far as the number of synchronized neighbors is constant through the lattice. What we have observed in our simulations is that these structures are attractors of the dynamics within different domains of the space of initial conditions. The corresponding fixed points are given by (13) changing ε by 4ε, 3ε, or 2ε. Nevertheless, these structures are not the only attractors, since more complicated patterns involving more than two different phases also can be built. Among of all these configurations, the "chessboard" one is the most relevant because it has the larger domain of attraction, although the relative weight depends on the parameters as well as on the system size, and it is the most stable one in the sense that small fluctuations break the other structures in its favor [24]. We want to point out that these phase-locked states are characteristic of lattice models with short-range interactions, since they have no counterpart with models of all-to-all pulse-coupled oscillators.

Finally, we believe that the tendency of our model either to synchronize or to form phase-locked structures allows us to go one step further in the current understanding of SOC phenomena. Middleton and Tang [13] noticed that SOC appears, in a uniformly driven model and a slightly different coupling, as a consequence of marginally stable phase locking between neighbors. We have shown how this marginal stability [which corresponds to the equality in the right hand side of (7)] is broken in favor of a synchronization or a stable phase locking depending on the driving rate and on the ERC. Thus, although further investigation along this line would be required, one can conjecture that SOC is critical in the sense that balances the tendency into one or another direction.

In this paper we have shown how it is possible to reduce a general population of biological oscillators to a simple model with constant ERC that allows analytical results for all-to-all coupling by means of a very easy transformation. Surprisingly, short-range simulations verify the same conditions for the synchronization that the long-range version of the model. Moreover, we also find new states of the system with no analogous states in the long-range case. These behaviors allow us to give some hints about the origin of SOC.

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