Polynomial planar vector field with two algebraic invariant curves

N. Sadovskaia[1] and R.Ramírez[2] I.

[1] **Departamento de Matemática Aplicada II**
Universitat Politècnica de Catalunya
natalia.sadovskaia@upc.edu

[2] **Departamento de Ingenería Informática**
Universitat Rovira i Virgili de Tarragona
rramirez@ etse.urv.es

Abstract

We construct a polynomial planar vector field of degree n with two invariant algebraic curves. For such a system we prove that the maximum number of algebraic limit cycles is

$$\frac{1}{2}(n-1)(n-2)+1.$$

1. Introduction

By definition a real (complex) planar polynomial differential system or simply a polynomial system is a differential system of the form

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases} \tag{1.1}$$

where the dependent variables x and y are real (respectively complex), the independent variable (time t) is real and functions P and Q are polynomial in x and y with real (complex) coefficients.

We denote by $n = max(deg\, P(x,y),\, deg\, Q(x,y))$ the degree of the polynomial system (1.1).

In this paper we are mainly interested in the polynomial differential system which possess a given pair of invariant algebraic curves.

We denote by $\mathrm{R}[x,y]$ the ring of real polynomials in x and y. An algebraic curve

$$g(x,y) = 0, \quad g \in \mathrm{R}[x,y]$$

is called an invariant algebraic curve of the system (1.1) if the following condition holds

$$P(x,y)\partial_x g(x,y) + Q(x,y)\partial_y g(x,y) = K(x,y)g(x,y)$$

for some real polynomial K of degree at most $n-1$. The function g is called a partial integral.

In what follows the derivatives of g with respect to x and y are designated as $\partial_x g$, $\partial_y g$.

Many papers are dedicated to the study of polynomial planar vector fields with algebraic invariant curves (see for instance [3, 9, 2, 15, 4, 10, 6, 8, 11]).

It is always helpful to look at this problem from another point of view. In this paper, we take an alternative viewpoint of starting with a given set of algebraic invariant curves and determining the form of the system which has such a set of invariant curves.

This approach was firstly developed by Eruguin in the paper "Construction of the whole set of ordinary differential equations with a given integral curve" published in 1952 [5]. In this article the author stated and solved the problem of constructing a planar vector field for which the given curve is its invariant. He obtained the following result:

Proposition 1.0.

The most general planar vector field for which

$$g(x, y) = 0$$

is its invariant curve is

$$\begin{cases} \dot{x} = \nu(x, y)\partial_y g + \ a(x, y) \\ \dot{y} = -\nu(x, y)\partial_x g + \ b(x, y) \end{cases} \tag{1.2}$$

where ν, a, b are functions:

$$\begin{cases} a(x, y)\partial_x g + \ b(x, y)\partial_y g = \Phi(x, y) \\ \Phi(x, y)|_{g(x,y)=0} = 0 \end{cases} \tag{1.3}$$

It is important to observe that Eruguin considered only one curve, moreover he didn't require that this curve was necessarily algebraic.

By applying the Eruguin ideas we state and study a new inverse problem in the theory of ordinary differential equations [11, 12]. We analyze the problem of constructing a planar vector field with $S \geq 1$ invariant curves, in particular we study the case when the given curves are algebraic. A systematic study of this problem was started in [11] In this paper we continue to further investigation of this problem. We study a polynomial planar vector field of degree n with only two invariant algebraic curves which we consider that are irreducible and non- singular. We complement the results given in [2,4] by showing that the maximum number of algebraic limit cycles which this system can admit is

$$\frac{1}{2}(n-1)(n-2) + 1.$$

We recall the main notations and definitions given in [11]

Let $\mathbf{v}$ be the vector field for which the algebraic curves

$$g_j(x, y) = 0, \quad j = 1, 2, ..., S, \quad S \geq 2$$

are its invariants, i.e.

$$dg(\mathbf{v}) = K_j(x, y)g_j(x, y), \quad K_j \in \mathrm{R}[x, y]. \tag{1.4}$$

Definition (Condition (A))

We say that the given set of invariant functions of the vector field $\mathbf{v}$:

$$g_1,\, g_2,\, ...,g_S, \quad S \geq 2.$$

satisfies the condition (A) if in this set there exist at least two functions, for example $g_1,\, g_2$, for which the algebraic curve

$$\{g_1,\, g_2\} \equiv \partial_x g_1 \partial_y g_2 - \partial_y g_1 \partial_x g_2 = 0, \tag{A}$$

does not contain the trajectories of the vector field $\mathbf{v}$. The following result is valid [13]:

Proposition 1.1

Suppose that $g_1,\, g_2,\, ...,g_S$, are functions which satisfy the condition (A). Then the most general planar vector field of degree n with these functions as partial integrals, is described by the system

$$\begin{cases} \dot{x} = \mu_1(x,y)g_1(x,y)\{g_2,\, x\} + \mu_2(x,y)g_2(x,y)\{x,\, g_1\} \equiv P(x,y) \\ \dot{y} = \mu_1(x,y)g_1(x,y)\{g_2,\, y\} + \mu_2(x,y)g_2(x,y)\{y,\, g_1\} \equiv Q(x,y) \end{cases} \tag{1.5}$$

with the following restrictions:

$$\mu_1 g_1\{g_2,\, g_j\} + \mu_2 g_2\{g_j,\, g_1\} + \mu_j g_j\{g_1,\, g_2\} = 0 \tag{1.6}$$

where $\mu_j, \quad j = 1, 2, ..., S$ are rational functions:

$$\begin{cases} \dot{g}_j = \{g_1,\, g_2\}\mu_j\, g_j \\ \{g_1,\, g_2\}\mu_j \in \mathrm{R}[x,y] \\ deg(\{g_1,\, g_2\}\mu_j) \leq n-1 \quad j = 1, 2, .., S \end{cases} . \tag{1.7}$$

In what follows a system composed by several systems; or a system with the same additional conditions, given by the indexed formulas, will be presented as the union of the correspondent expressions; i.e. if these expression are indexed as (i,j), (k,l),..., (m,n) then a new system which satisfies all theses expressions will be designated as (i,j)+ (k,l)+... +(m,n).

2. The Poincaré and Darboux problems for planar vector field with two invariant algebraic curves

In this section we solve the Poincaré problem (the existence of an upper bound for the degree of invariant algebraic curve) and Darboux problem (the existence of Darboux first integral) for the polynomial planar vector field of degree n with two invariant algebraic curves:

$$\mathbf{v} = g_1\mu_1\{\ ,g_2\} + g_1\mu_1\{g_1,\ \}, \tag{2.0}$$

where

$$\{g,\ \} \equiv \partial_x g \partial_y - \partial_y g \partial_x,$$

which generates the differential system (1.5).

First of all we shall prove the following proposition which gives a solution the Poincaré problem for the planar vector field with two invariant curves.

Proposition 2.1

Suppose that (2.0) is not Darboux's integrable and such that

$$\begin{cases} \mu_j \in \mathrm{R}[x,y] \\ deg\mu_j = m_j \leq n-1-r,\ j=1,2, \\ r = deg\{g_1,\ g_2\} \end{cases} \tag{2.1}$$

Let l, k are the degree of the polynomials g_1, g_2 respectively, then

$$2 \leq l+k \leq n+1 \tag{2.2}$$

In fact, let us assume that the given polynomials g_1, g_2 admit the representations

$$\begin{aligned} g_1(x,y) &= x^{l_1}y^{l_2} + ..., \quad l_1+l_2 = l \\ g_2(x,y) &= x^{k_1}y^{k_2} + ..., \quad k_1+k_2 = k \end{aligned}$$

By inserting these expressions into (2.0) we obtain

$$\mathbf{v} = \big((\mu_2 l_2 - \mu_1 k_2)x^{l_1+k_1}y^{l_2+k_2-1} + ...\big)\partial_x + \big((\mu_1 k_1 - \mu_2 l_1)x^{l_1+k_1-1}y^{l_2+k_2} + ...\big)\partial_y$$

therefore, by considering that

$$max(deg(\mathbf{v}(x), deg(\mathbf{v}(y)) = n$$

we deduced(2.2).

If

$$\begin{cases} \mu_2 l_2 - \mu_1 k_2 = 0 \\ \mu_1 k_1 - \mu_2 l_1 = 0 \end{cases}$$

then the constructed vector field is Darboux's integrable as follows from the following proposition which gives a solution the Darboux problem for vector field with (2.0).

Proposition 2.2

The vector field (2.0) is Darboux's integrable iff

$$\begin{cases} \mu_1 = a(x,y)B_0 \\ \mu_2 = -a(x,y)A_0 \end{cases} \tag{2.3}$$

where a is an arbitrary rational function, A_0, B_0 are nonzero constants.

Clearly that if (2.3) holds then the vector field $\mathbf{v}$ takes the form

$$\mathbf{v} = a\, g_1 g_2 \{\ln\left(g_1^{A_0}\, g_2^{-B_0}\right), \qquad \}$$

On the other hand if

$$F(x, y) = g_1^{\lambda_1}\, g_2^{\lambda_2}$$

is a first integral of (1.5) then

$$\lambda_1\mu_1 + \lambda_2\mu_2 = 0$$

and therefore we obtain (2.3), with $B_0 = \lambda_2$, $A_0 = \lambda_1$.

Corollary 2.1

Suppose the given functions g_1 and g_2 are complex function:

$$g_1 = U + i\,V = |g_1|e^{i\Theta}, \quad g_2 = U - i\,V = |g_1|e^{-i\Theta}$$

where $|g_1|^2 = U^2 + V^2$

Then the first integral F :

$$F(x, y) = |g_1|^a e^{b\Theta}$$

where a, b are real constants.

The proposition (2.2) can be generalized for the case when the number of the given invariant curves is greater than two [11].

Proposition 2.3

The vector field (1.5)+(1.6) is Darboux integrable iff

$$g_j\mu_j = \nu \prod_{k=1}^{S} g_k\{lnF(x, y),\, g_j\}, \quad j = 1, 2$$

$$F(x, y) = \prod_{k=}^{S} g_k^{\lambda_k}$$

where ν is an arbitrary rational function and $\lambda_j, \quad j = 1, 2, .., S$ are constants.

With the following example we show that, in general, for a Darboux integrable polynomial vector field there does not exist an upper bound for the degree of invariant algebraic curves.

Example 2.1

Let us suppose that the given partial integrals are such that

$$\begin{cases} g_1 = x \\ g_2 = y \\ g_3 = y^m + g(x, y) \end{cases}$$

where m is an arbitrary natural number and g is a polynomial of degree $k + 1$ in x, y.

By applying the proposition 2.3 for the given partial integrals, we obtain that

$$\begin{cases} \mu_1 = -\nu((\lambda_2 + m\lambda_3)y^m + \lambda_2\, g(x, y) + \lambda_3 y\, \partial_y g(x, y)) \\ \mu_2 = \nu\,(\lambda_1(y^m + g(x, y)) + \lambda_3 x\, \partial_x g(x, y)) \end{cases}$$

By choosing

$$\lambda_1 = 0,\ \lambda_2 = -m,\ \lambda_3 = 1$$

the functions $\mu_1,\ \mu_2$ take the form

$$\begin{cases} \mu_1 = \nu\, y g_3(x,y)\{lnF(x,y),\ x\} \\ \mu_2 = \nu\, y g_3(x,y)\{lnF(x,y),\ y\} \\ F(x,y) = \dfrac{g_3(x,y)}{y^m} \end{cases}$$

In particular if

$$\begin{aligned} & g(x,y) = x^k(p_0\, y + p_1 x\, p_2) \\ & \nu = x^{-k} \end{aligned}$$

we obtain the quadratic vector field

$$\mathbf{v} = (p_0(m-1)\, y + p_1 m\, x + p_2 m) x\partial_x + (p_0 k\, y + p_1(k+1)\, x + p_2\, k) y\partial_y$$

which admits as invariant algebraic curve the curve $y^m + g(x,y) = 0$ of degree $l = max(m,\ k+1)$

Now we present three examples of planar vector field with two invariant algebraic curves. These examples are important for the further statements and results given in this paper.

Example 2.2

In this example we construct a non Darboux's integrable polynomial vector field with two invariant algebraic curves of the maximum degree (equal to n) and minimum degree (equal to 1) (see proposition 2.1).

Let us consider the following functions

$$\begin{aligned} & g_1(x,y) = x - a, \\ & g_2(x,y) = x^n + G(x,y) \end{aligned}$$

where a is a real parameter and G is a polynomial of degree $n-1:$

$$\begin{cases} \{g_1,\ g_2\} = \partial_y G(x,y) \\ deg(\ \partial_y\, G(x,y)) = n-2 \end{cases}$$

The polynomial vector field of degree n for which the above functions are its partial integrals is the following

$$\begin{cases} \dot{x} = (Ax + By + C)(x-a)\partial_y g_2(x,y) \\ \dot{y} = -(Ax + By + C)(x-a)\partial_x g_2(x,y) + (\beta + n((Ax + By + C)) g_2(x,y) \end{cases} \tag{2.4}$$

where $A,\ B,\ C,\ \beta$ are real parameters: $B \neq 0$.

By considering that in this case the functions μ_1, μ_2 are such that

$$\mu_1 = (Ax + By + C)\partial_y G, \quad \mu_2 = \big(\beta + n(Ax + By + C)\big)\partial_y G,$$

from the proposition 2.2 we deduced that the constructed system (2.4) is non Darboux integrable iff $\beta \neq 0$.

If $\beta = 0$ then from (2.3) we obtain that

$$\begin{cases} a(x, y) = (Ax + By + C)\partial_y G \\ B_0 = -1, \quad A_0 = -n \end{cases}$$

hence the function

$$F(x, y) = \frac{x^n + G(x, y)}{(x - a)^n}$$

is a Darboux first integral of (2.4).

We observe that if $B = 0$ then the system (2.4) admits three partial algebraic integrals. After some calculations we can prove that

$$F(x, y) = |x^n + G(x, y)|^{C+Aa}|x - a|^{\beta-n(Aa+C)}|Ax + C|^{-A\beta}$$

is the first integral of (2.4).

Example 2.3

We study the family of Darboux integrable cubic planar vector field for which the given conics are its invariants:

$$\mathbf{v} = y(a + b\,x^2 + c\,y^2)\partial_x + x(\alpha + \beta\,x^2 + \gamma\,y^2)\partial_y$$

where a, b, c, α, β, γ are real constants:

$$b\gamma - c\beta \neq 0$$

the differential equations generated by the above vector field are

$$\begin{cases} \dot{x} = y(a + b\,x^2 + c\,y^2) \\ \dot{y} = x(\alpha + \beta\,x^2 + \gamma\,y^2) \end{cases} \tag{2.5}$$

In the all assertions below we shall analyze the case when $c \neq 0$. The case when $c = 0$ is easy to analyze.

The following proposition is valid

Proposition 2.4

Let g_1, g_2 are the functions:

$$g_j(x, y) = \nu_j(x^2 - \lambda_0) - y^2 + \lambda_1, \quad j = 1, 2$$

where λ_0, λ_2, ν_1, ν_2 are constants:

$$\begin{cases} \lambda_0 = \dfrac{\gamma\, a - \alpha\, c}{b\gamma - c\beta}, \quad \lambda_1 = \dfrac{\alpha\, b - \beta\, a}{b\gamma - c\beta}, \\ \nu_1 = \dfrac{\gamma - b}{2c} + \sqrt{(\dfrac{\gamma - b}{2c})^2 + \dfrac{\beta}{c}}, \quad \nu_2 = \dfrac{\gamma - b}{2c} - \sqrt{(\dfrac{\gamma - b}{2c})^2 + \dfrac{\beta}{c}} \\ \nu_1 - \nu_2 \neq 0 \end{cases}$$

The following relations hold:

$$\begin{cases} dg_j(\mathbf{v}) = 2x\, y\, (\gamma - \nu_j\, c) g_j, \quad j = 1, 2 \\ \{g_1,\, g_2\} = 4x\, y\, (\nu_1 - \nu_2) \end{cases} \tag{2.6}$$

The proof is easy to obtain after some calculations.

From (2.6) we can observe that the algebraic curves

$$\nu_j (x^2 - \lambda_0) - y^2 + \lambda_1 = 0, \quad j = 1, 2$$

are invariant curves (singular solutions) of $\mathbf{v}$ and do not satisfies the condition (A).

By compare (2.6) with (1.7) we deduced that

$$2\mu_j = \frac{\gamma - \nu_j c}{\nu_1 - \nu_2}, \quad j = 1, 2$$

as a consequence, in view of the proposition 2.2, we obtain that the given vector field is Darboux integrable with F :

$$F(x, y) = \frac{(\nu_1 (x^2 - \lambda_0) - y^2 + \lambda_1)^{b + \nu_1 c}}{(\nu_2 (x^2 - \lambda_0) - y^2 + \lambda_1)^{b + \nu_2 c}},$$

here we use the relation

$$c(\nu_1 + \nu_2) = \gamma - b.$$

Now we shall study the case when the roots ν_1, ν_2 are complex numbers.

By introducing the notations

$$\gamma - b = 2cq, \quad \gamma + b = 2cr$$

we obtain that

$$\nu_1 = q + i\, p, \quad \nu_2 = q - i\, p, \quad p^2 = -4\beta\, c - q^2,\, p\, > 0$$

The system (2.5) takes then the form (we put $c = 1$) :

$$\begin{cases} \dot{x} = y(a + (r - q)\, x^2 + y^2) \\ \dot{y} = x(\alpha - (p^2 + q^2)\, x^2 + (r + q) y^2) \end{cases} \tag{2.7}$$

By applying the corollary 2.1, by considering that in this case

$$g_1(x,y) = q(x^2-\lambda_0) - y^2 + \lambda_1 + ip(x^2-\lambda_0),$$

we obtain that the first integral F takes the form

$$F(x,y) = \Big((y^2-\lambda_1-q(x^2-\lambda_0))^2 + p^2(x^2-\lambda_0)^2\Big) exp\Big(2r\, arctg\frac{p(x^2-\lambda_0)}{y^2-\lambda_1-q(x^2-\lambda_0)}\Big)$$

Clearly that the singular solutions of (2.7) are

$$\begin{aligned} y^2-\lambda_1 &= 0 \\ x^2-\lambda_0 &= 0. \end{aligned}$$

For $r=0$ we obtain that the system (2.7) is the Hamiltonian cubic differential system with Hamiltonian

$$H_0 = y^4 - 2qx^2y^2 + (q^2+p^2)x^4 + 2(q\lambda_1 - (q^2+p^2)\lambda_0)x^2 + 2\,y^2(q\lambda_0-\lambda_1)$$

If in (2.7) we introduce the Hamiltonian H :

$$\begin{cases} H = \dfrac{1}{4}(c\,y^4 - 2b\,x^2\,y^2 + \beta\,y^4) + \dfrac{1}{2}(a\,y^2 - \alpha\,x^2) \\ H|_{r=0} = F(x,y)|_{r=0} \end{cases} \tag{2.8}$$

then we can rewritten it as follows

$$\begin{cases} \dot{x} = \partial_y H(x,y) \\ \dot{y} = -\partial_x H(x,y) + 2cr\,x\,y^2 \end{cases}$$

An interesting fact is that the hamiltonian of the type (2.8) appears in particular when we study the mechanical system with two degrees of freedom near a closed trajectory or in the neighborhood of an equilibrium position [1].

Corollary 2.2

The solutions of the differential equations (2.5) can be represented as follows

$$\begin{cases} x^2 = \lambda_0 + X(\tau, x_0, y_0) \\ y^2 = \lambda_1 + Y(\tau, x_0, y_0) \\ t = t_0 + \displaystyle\int_0^\tau \frac{d\tau}{\sqrt{(\lambda_0 + X(\tau,x_0,y_0))(\lambda_1 + Y(\tau,x_0,y_0)}} \end{cases} \tag{2.9}$$

where X, Y are solutions of the linear differential equation of the second order with constants coefficients

$$T^{''} - (\gamma+b)T^{'} + (b\gamma - c\beta)T = 0, \quad {}^{'} \equiv \frac{d}{d\tau} \tag{2.10}$$

In fact, it is easy to prove that the functions X, Y are solutions of the equations

$$\begin{cases} X^{'} = b\,X + c\,Y \\ Y^{'} = \beta\,X + \gamma\,Y \end{cases}$$

and then by taking the second derivatives of X and Y we get (2.10).

It is interesting to observe that if we introduce the complex coordinate in the plane R^2 :

$$z = x + iy, \quad z_* = x - i\,y$$

then differential equation

$$\dot{z} = i\,(a_{10}z + a_{01}z_* + \sum_{j+k=3} a_{jk}z^j\,z_*^k) \tag{2.11}$$

is Darboux integrable, where $a_{jk},\ j,k = 0,2,3$ are real constants.

In fact, the equation (2.11) is equivalent to the cubic planar system of the type (2.5):

$$\begin{cases} \dot{x} = y(a_{01} - a_{10} + (a_{12} - a_{21} + 3(a_{03} - a_{30})\,x^2 + (a_{12} - a_{21} + a_{30} - a_{03})\,y^2) \\ \dot{y} = x(a_{01} + a_{10} + (a_{12} + a_{21} + a_{03} + a_{30})\,x^2 + (a_{12} + a_{21} - 3(a_{30} + a_{03}))y^2) \end{cases}$$

Now we shall study the particular case of (2.5) when $\nu_1,\ \nu_2$ are real constants and the vector field is such that

$$\begin{cases} \dot{x} = y(\lambda\,b + p + x^2(\lambda + b - 2a) + y^2(\lambda - b)) \\ \dot{y} = x(-\lambda a - p - x^2(\lambda - a) - y^2(\lambda + a - 2b)) \end{cases} \tag{2.12}$$

where λ, b, a, p are real parameters: $b - a\ > 0$

The first integral F in this case is the following

$$F(x,y) = \frac{(y^2 + x^2 + \lambda)^2}{(\lambda - a)x^2 + (\lambda - b)y^2 + \frac{1}{2}(\lambda^2 + p)}$$

and

$$\nu_1 = -1, \quad \nu_2 = \frac{a - \lambda}{\lambda - b}.$$

Clearly the singular solutions of (2.7) are

$$g_1(x,y) \equiv y^2 + x^2 + \lambda = 0$$
$$g_2(x,y) \equiv (\lambda - a)x^2 + (\lambda - b)y^2 + \frac{1}{2}(\lambda^2 + p) = 0$$

The all nonsingular solutions of the differential system (2.12) study are algebraic curves of the form

$$(x^2 + y^2)^2 + A(\lambda\,, K, p)\,x^2 + B(\lambda\,, K, p)\,y^2 + P(\lambda\,, K, p) = 0 \tag{2.13}$$

where $F(x, y) = K$ are the level curves,

$$\begin{cases} A((\lambda\,, K, p) = 2(\lambda - \dfrac{K}{2}(\lambda - a)) \\ B(\lambda\,, K, p) = 2(\lambda - \dfrac{K}{2}(\lambda - b)) \\ P(\lambda\,, K, p) = \lambda^2 - \dfrac{1}{2}K(\lambda^2 + p) \end{cases}$$

The solutions of the system (2.12) can be represented as follows (see corollary 2.2)

$$\begin{cases} x^2 = \dfrac{\lambda^2 - 2b\,\lambda - p}{2(b-a)} + \dfrac{(b-\lambda)\,g_1(x_0, y_0)}{b-a} e^{(b-a)\tau} + \dfrac{g_2(x_0, y_0)}{b-a} e^{2(b-a)\tau} \\ y^2 = -\dfrac{\lambda^2 - 2a\,\lambda - p}{2(b-a)} + \dfrac{(\lambda - a)\,g_1(x_0, y_0)}{b-a} e^{(b-a)\tau} - \dfrac{g_2(x_0, y_0)}{b-a} e^{2(b-a)\tau} \\ t = t_0 + (b-a) \displaystyle\int_0^\tau \frac{d\tau}{\sqrt{R_4(e^{(b-a)\tau)}}} \end{cases} \tag{2.14}$$

where R_4 is a polynomial:

$$R_4(u) = -\,(g_2(x_0, y_0)\,u^2 + (b-\lambda)\,g_1(x_0, y_0)\,u + \frac{1}{2}(\lambda^2 - 2b\,\lambda - p))$$
$$(g_2(x_0, y_0)\,u^2 + (a-\lambda)\,g_1(x_0, y_0)\,u + \frac{1}{2}(\lambda^2 - 2a\,\lambda - p))$$

Now we study a particular case when

$$b = -a = 1, \quad p = c^4 - 1 \tag{2.15}$$

In this case the equation (2.12), in complex coordinates, takes the form (see (2.11))

$$\dot{z} = i[(1-c^4)\,z - \,z^3 + \lambda\,(z_* - z^2 z_*)] \tag{2.16}$$

The trajectories of this equation are given by the formula (2.13)+(2.15). If $K = 2$ we obtain the lemniscata with two foci ($(1, 0)$, $(-1, 0)$) and with the radius c :

$$(z^2 - 1)(z_*^2 - 1) = c^4$$

If the radius $c < 1$ then we get the Cassini ovals, if $c = 1$ the Bernoulli lemniscate and if $c > 1$ the Booth lemniscate.

In particular if in (2.16) $\lambda = 0$ we obtain the differential equation

$$\dot{z} = i(1 - c^4)\,z - i\,z^3 \tag{2.17}$$

From (2.13)+(2.15) we deduce that all the trajectories of (2.17) belong to the one-parametric family of lemniscates

$$(z^2 - \frac{K}{2})(z_*^2 - \frac{K}{2}) = \frac{K}{2}(\frac{K}{2} + c^4 - 1)$$

It is easy to observe that for $c \neq 1$ the equilibrium points of (2.16) are isochronous centres and the all solutions can be obtained from (2.14) by considering that

$$\tau = \begin{cases} ln\,(1 - 2D\cos 2\,pt + D^2)^{\frac{1}{8}} & \text{if } c \neq 1 \\ ln\,(D^2 + t^2)^{-\frac{1}{8}} & \text{if } c = 1 \end{cases} \tag{2.18}$$

where D is a real constant.

It is important to note that from the Eruguin result (see proposition 1.0) follows that the cubic vector field for which the invariant curve is the lemniscate with foci $(1, 0)$, $(-1,\,0)$ and radius c is the following:

$$\begin{cases} \dot z = i[(1 - c^4)\,z - \,z^3 + \lambda\,(z_* - z^2 z_*)] + k(z^3 - 2\,z) \\ k\,(1 - c^4) = 0, \quad c,\,\lambda,\,k \in \mathrm{R} \end{cases} \tag{2.19}$$

For $k \neq 0$ this equation is non-Darboux integrable.

The complete bifurcation diagram in the hemiplane $(c > 0\,, \lambda)$ of the equation (2.12) is given in [14].

Example 2.4

From the proposition 2.3 we deduce that the cubic vector field with two invariant conics

$$g_j(x, y) = y^2 + a_j\,x^2 + b_j\,x + c_j\,y + d_j = 0 \quad j = 1, 2$$

is Darboux integrable if $a_1 - a_2 \neq 0$ [11].

In this example we shall construct the non-Darboux integrable cubic vector field for which the given circumferences

$$g_j = |z - A_j|^2 - r_j^2 = 0 \quad j = 1, 2 \tag{2.20}$$

are its invariant curves, where

$$A_1 - A_2 \neq 0.$$

For simplicity we shall take

$$\begin{cases} A_1 = 0, \quad A_2 = 1 \\ r_1 = r_2 = r \end{cases} \tag{2.21}$$

The following proposition holds:

Proposition 2.5

The set of cubic differential equation depending on the parameter r

$$\begin{aligned} \dot z = i\,z\Big((\alpha\,z + \alpha_*\,z_* - 1)(|z-1|^2 - r^2) - (\alpha\,z + \alpha_*\,z_*)(|z|^2 - r^2)\Big) + \\ i(\alpha\,z + \alpha_*\,z_*)(|z|^2 - r^2) \end{aligned} \tag{2.22}$$

has as invariant curves the circumferences (2.20)+(2.21), where $\alpha = \frac{1}{2}(1+i)$, $\alpha_* = \frac{1}{2}(1-i)$.

In fact it is easy to show that along the solutions of (2.22) we have that

$$\dot{g_1} = (\alpha\, z + \alpha_*\, z_*)g_1$$
$$\dot{g_2} = (\alpha\, z + \alpha_*\, z_* - 1)g_2$$

The bifurcation values of the parameter r are

$$r = \frac{1}{2}, \quad r = \frac{\sqrt{2}}{2}, \quad r = \sqrt{2\sqrt{2} - 2}, \quad r = 1, \quad r = \frac{3}{2}.$$

From the bifurcation diagrams of (2.22) given in the figure below we obtain that for the values of the parameter $r < \frac{1}{2}$ the circumferences (2.20)+(2.21) are isolated periodical solutions, i.e., algebraic limit cycles.

It is interesting to observe that this result can be generalized for the polynomial planar vector field of degree n with $n-1$ invariant circumferences [11].

3. Maximum number of algebraic limit cycles for the polynomial system with two invariant algebraic curves

In this section we prove that the maximum numbers of algebraic limit cycles for non-Darboux integrable polynomial system system of degree n with two irreducible algebraic invariant curves is

$$\frac{1}{2}(n-1)(n-2) + 1.$$

Let suppose the vector field (2.0)+(2.1) is a non-Darboux integrable, polynomial vector field of degree n, and $A(n, 2)$, is the maximal number of algebraic limit cycles which (2.0)+(2.1) can admits and f is the function:

$$f(u_1, u_2) = \frac{1}{2}(u_1 - 1)(u_1 - 2) + \frac{1}{2}(u_2 - 1)(u_2 - 2) + \alpha_1 + \alpha_2 \tag{3.1}$$

where

$$\alpha_j = \begin{cases} 1, \text{ if } u_j > 1 \\ 0, \text{ if } u_j = 1, \quad j = 1, 2 \end{cases}$$

The following assertion takes place.

Proposition 3.1

The following inequality holds

$$f(1, n-1) \leq A(n, 2 \leq f(1, n) \quad \forall n > 2 \tag{3.2}$$

In fact, to obtain the upper bound $f(1, n)$ we solve the following problem

Problem 3.1

Determine the maximum and minimum values of the function (3.1) on the compact domain

$$\mathcal{D} = \{(u_1,\, u_2) \in \mathrm{R}^2 : 2 \leq u_1 + u_2 \leq n + 1, \quad n \in \mathrm{N},\, n \geq 3\}$$

i.e., determine

$$extremum f(u_1,\, u_2)$$
$$(u_1, u_2) \in \mathcal{D}$$

By solving this problem we see that extremum values of f are achieved in the points $(1,\, n)$, $(n,\, 1)$ and $(\frac{n+1}{2},\, \frac{n+1}{2})$ respectively. The values of f at these points are respectively equal:

$$f(1,\, n) = f(n,\, 1) = \frac{1}{2}(n-1)(n-2) + 1$$
$$f(\frac{n+1}{2}, \frac{n+1}{2}) = \frac{1}{4}(n-1)(n-3) + 2$$

clearly $f(\frac{n+1}{2},\, \frac{n+1}{2}) \;<\; f(1, n-1)\,, \forall n \geq 4.$

For $n = 3$ we obtain that

$$f(3,1) = f(1,3) = f(2,2) = 2$$

On the other hand, from the Harnack theorem we see that the number of connected components of the algebraic curve of degree m is $\frac{1}{2}(m-1)(m-2)+1$, hence

$$f(m_1,\, m_2) = \frac{1}{2}(m_1-1)(m_1-2) + \frac{1}{2}(m_2-1)(m_2-2) + \alpha_1 + \alpha_2$$

can be interpreted as the maximum number of the compact components of the algebraic curves of degree m_1 and m_2 respectively.

It is clear that the maximum number of the algebraics limit cycles for the vector field (2.0)+(2.1), by applying the solution of the Poincaré problem (see proposition 2.1), is such that

$$A(n,2) \leq f(m_1,\, m_2) \leq max|_{(u_1,u_2)\in\mathcal{D}} f(u_1,\, u_2) = f(1,n)$$

Hence, the maximum number of algebraics limit cycles for a non-Darboux integrable planar vector field we can be achieved in the polynomial vector field with two invariant algebraic curves

$$g_1 = x - a = 0,$$
$$g_2(x,y) = 0,$$

where $deg\, g_2 = n$, i.e. in the polynomial differential system

$$\begin{cases} \dot{x} = \mu_1(x,y)(x-a)\partial_y g_2(x,y) \\ \dot{y} = -\mu_1(x,y)(x-a)\partial_x g_2(x,y) + \mu_2(x,y)g_2(x,y) \end{cases} \tag{3.3}$$

where μ_j, $j = 1, 2$ are polynomials such that

$$deg(\mu_j(x,y)) \leq 1,$$

We choose the parameter a in such a way that the straight line $x - a = 0$ does not intersected the ovals of the algebraic curve $g_2(x,y) = 0$.

The system

$$\begin{cases} \dot{x} = (x-a)\partial_y g_2(x,y) \\ \dot{y} = (x-a)\partial_x g_2(x,y) + y\, g_2(x,y) \end{cases} \tag{3.4}$$

where $deg\, g_2 = n-1$ was studied in [Dolov, Bautin]. The authors proved that this system admits (by choosing g_2 properly) $f(1, n-1)$ algebraic limit cycles.

Hence we obtain the lower bound in the relation (3.2).

Proposition 3.2

The maximum numbers of algebraic limit cycles which can admit a polynomial vector field of degree $n \geq 3$ with two invariant algebraic curves is

$$A(n,2) = f(1,n) = \frac{1}{2}(n-1)(n-2) + 1$$

In fact, in the paper [Kharlamov et al] the authors claim the existence of a curve with the maximal possible of connected components in the situation when the Newton polygon is prescribed, and this number is just the number of interior points in the polygon plus one. In particular for the curve

$$g_2(x,y) = x^n + G(x,y) = 0, \quad deg\, G(x,y) = n-1 \tag{3.5}$$

this number is equal to $\frac{1}{2}(n-1)(n-2)+1$

On the other hand, from the example 2.2 we obtain a non-Darboux integrable polynomial vector field of degree n with invariant algebraic curve (3.5).

By applying the above result we can determine the parameters A, B, C, β, a in such a way that the vector field (2.4) admits $\frac{1}{2}(n-1)(n-2)+1$ algebraic limit cycles.

Corollary 3.1

For the cubic vector field with two invariant algebraic curves we have that the maximum number of algebraic limit cycles (two) can be achieved if we have two invariant circumferences, i.e.

$$A(3,2) = f(2,2) = 2$$

The proof follows from the example 2.5.

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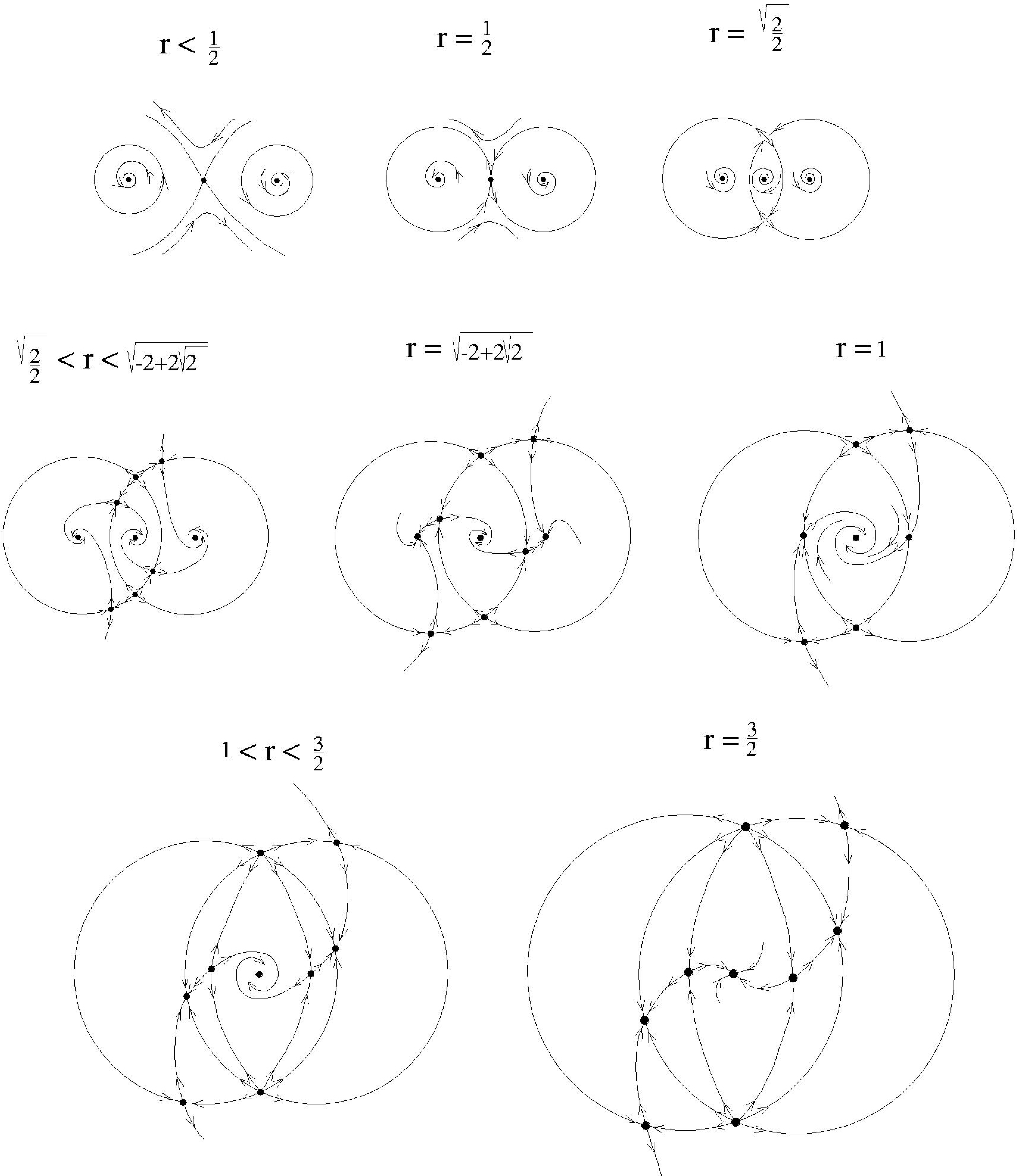
$r < \frac{1}{2}$
$r = \frac{1}{2}$
$r = \sqrt{\frac{2}{2}}$
$\sqrt{\frac{2}{2}} < r < \sqrt{-2+2\sqrt{2}}$
$r = \sqrt{-2+2\sqrt{2}}$
$r = 1$
$1 < r < \frac{3}{2}$
$r = \frac{3}{2}$