ON THE DYNAMIC OF NONHOLONOMIC SYSTEMS

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Abstract We proposed a mathematical model (*A-model*) to describe the behavior of the nonholonomic system. This approach is based on the generalized Hamiltonian Principle, with nonzero transpositional relations.

Key words. Nonholonomic system, nonintegrable constraints, nonlineal constraints, Chetaev conditions, equation of motions, principles.

1.INTRODUCTION.

First we introduce the main concept and notations which we can find in particular in [Arnold et all].

To describe the motion of mechanical system one uses a variety of mathematical models which are based on different *Principles*- laws of motion .

The simplest and most important model of motion of real bodies is

1.1 NEWTONIAN MECHANICS

which describes the motion of a free system of interacting point masses in three-dimensional Euclidian space E^3. This set is also known as the *configuration space* of the point s. The pair (s, v) where v is the velocity of the point, is called the sate of s, and the space $E^3 \times \mathrm{R}^3\{v\}$ is the *phase space.*

Definition 1.1

The pair $(m\,, \mathbf{r})$, where $\mathbf{r}$ is the position vector of the points is called a *material point* (or particle) of mass m.

Now we consider the more general case in which n particle

$$(m_1, \mathbf{r}_1), (m_2, \mathbf{r}_2), ...(m_n, \mathbf{r}_n),$$

are moving in E^3. The set

$$E^{3n} = E^3(\mathbf{r}_1) \times E^3(\mathbf{r}_2) \ldots \times E^3(\mathbf{r}_n)$$

is called the configuration space of this system. In this case it is necessary to exclude collisions of points, we must reduce E^{3n} by removing the diagonal

$$\bigcup_{i<j} \{s_i = s_j\}.$$

The set

$$E^{3n} \times \mathrm{R}^{3n}\{\mathbf{v}\},$$

where $\mathbf{v} = (\dot{\mathbf{r}}_1,\, \mathbf{r}_2 \,...,\dot{\mathbf{r}}_n)$ is the vector velocity of the system, is called the phase space of the system.

The Principle of Determinacy. This principle affirm that the state of the mechanical system at any fixed moment of time uniquely determines of all of its motion. Actually this principle is equivalent to the existence of the Newton equations

$$m_j\ddot{\mathbf{r}}_j = \mathbf{F}_j(t, \mathbf{r}, \dot{\mathbf{r}}), \quad \mathbf{r} = (r_1,\, r_2,\, ...,r_n)$$

where

$$\mathbf{F}_j : \mathrm{R}\{t\} \times \mathrm{R}^{3n}\{\mathbf{r}\} \times \mathrm{R}^{3n}\{\dot{\mathbf{r}}\} \to \mathrm{R}^{3n}$$

is a smooth vector function, called the force acting on the point of mass m_j.

The following fundamental model in mechanics is

1.2 LAGRANGIAN MECHANICS.

Lagrangian model, describe the behavior of the non-free motion of n particle

$$(m_1, \mathbf{r}_1), (m_2, \mathbf{r}_2), ...(m_n, \mathbf{r}_n),$$

the *constraints* are defined by smooth manifold $\mathcal{Q}$ embedded in the configuration space of the free system E^{3n}. These constraints allow only those motions for which

$$(\mathbf{r}_1(t), \mathbf{r}_2(t), ...\mathbf{r}_n(t)) \in \mathcal{Q} \quad \forall t.$$

If known forces $\mathbf{F}_1, \mathbf{F}_2, ..., \mathbf{F}_n$ act on the given points then the equations

$$\sum_{j=1}^{n} < m_j\ddot{\mathbf{r}}_j - \mathbf{F}_j, \xi_j >= 0 \tag{1.1}$$

where ξ_j is an arbitrary vector tangent to $\mathcal{Q}$, is called the *General Equation of Dynamics,* or the *d'Alembert-Lagrange Principle.* For the free systems of points, the vectors ξ_j are completely arbitrary, and hence equation (1.1) is equivalent to the system of Newton's equations.

Definition 1.2

Let $(x^1, x^2, .., x^N)$ be a local coordinates on $\mathcal{Q}$. A *Lagrangian system* on the smooth manifold $\mathcal{Q}$ is specified by a single function

$$L : T\mathcal{Q} \times \Delta \mapsto \mathrm{R},$$

where $\Delta \subset \mathrm{R}$. We call the point $x = (x^1,\, x^2,\, ..., x^N) \in \mathcal{Q}$ configuration of the system, and each tangent vector $\dot{x} \in T_x\mathcal{Q}$ it called a velocity at x. The natural number N is called the degree of freedom of the system.

The manifold $\mathcal{Q}$ and tangent bundle $T\mathcal{Q}$ is called the *configuration space* and *phase space* respectively.

Let a_1, a_2 be two point of $\mathcal{Q}$. We call a *path* from a_1 to a_2, starting at time t_1 and ending at time t_2 where $t_1,\, t_2 \in \Delta$ any C^∞ map

$$\omega : \Delta \to \mathcal{Q}$$

such that $\omega(t_1) = a_1, \quad \omega(t_2) = a_2$.

We let by Ω the all such paths.

Definition 1.3

Let $x : [t_1,\, t_2] \to \mathcal{Q}$ be a smooth path belonging to Ω.

At each time $t : t \in [t_1,\, t_2]$ the set of numbers

$$E_k(L) \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} - \frac{\partial L}{\partial x^k}$$

are called the *Lagrangian derivative.*

In the assertions below we shall exploding the fundamental properties of the Lagrangian derivatives, which we can established by straightforward calculation.

1.

$$E_k(L) \equiv E_k(L + \frac{df(x)}{dt})$$

for arbitrary smooth function f on $\mathcal{Q}$.

2. If we pass to new coordinates x^* by the rule $x = x(x^*)$ then

$$E_k(L(x, \dot{x}, t) = {J^*}^{-1} E_k(L^*(x^*,\, \dot{x}^*, t))$$

, where $J^* = (\frac{\partial x}{\partial x^*})$ is the nonsingular Jacobi matrix.

Theorem (Lagrange)

The functions $x(t)$ which describe the motion of the constrained system satisfy the equation

$$E_k(T) = Q_k, \quad k = 1, 2, .., N$$

where T is the kinetic energy of the system and $Q = (Q_1,\, Q_2, .., Q_N)$ is the *generalized forces*

For the case when

$$\sum_{j=1}^{N} Q_k dx^k = dU$$

then we can reexpress the above equations in the *Lagrangian form:*

$$E_k(L) = 0, \quad L = T + U, \quad k = 1, 2, .., N.$$

Definition 1.4

The functional

$$F(\omega) = \int_{t_1}^{t_2} L(\dot{\omega}(t),\, \omega(t))dt$$

is called the action

Hamiltonian Principle

The path $\omega \in \Omega$ is called motion of the Lagrangian system $(L, \mathcal{Q})$ if it is a critical point of the action, i.e., is a solution of the Lagrangian equations.

Another fundamental model in mechanics is

1.3 NONHOLONOMIC LAGRANGIAN MECHANICS.

Definition 1.5

A *nonholonomic Lagrangian system* is a triplet $(\mathcal{Q}, L, \mathcal{D})$ where $\mathcal{Q}$ is a smooth N-dimensional manifold called the *configuration space*, $L : T\mathcal{Q} \to \mathrm{R}$ is a smooth function called the *Lagrangian* and $\mathcal{D} \subset T\mathcal{Q}$ is a submanifold given locally by equations

$$\mathcal{D} = \{\dot{x} \in T\mathcal{Q} | f_\beta(x, \dot{x}, t) = 0, \quad \beta = 1, 2, ..., M\}$$

with linearly independent covectors

$$\partial f_\alpha \equiv (\frac{\partial f_\alpha}{\partial \dot{x}^1}, \frac{\partial f_\alpha}{\partial \dot{x}^2} ..., \frac{\partial f_\alpha}{\partial \dot{x}^N}), \quad \alpha = 1, 2, .., M$$

The constraints allow only those paths $\tilde{\omega} : \Delta \to \mathcal{Q}$ for which

$$(\tilde{\omega}, \dot{\tilde{\omega}}(t)) \in \mathcal{D}, \quad \forall t \in \Delta.$$

In this paper we shall consider mainly linear autonomous constraints, i.e., when the functions f_α are such that

$$f_\alpha = \sum_{k=1}^{N} a_{\alpha k}(x)\dot{x}^k$$

Hertz has classified Lagrangian system with linear constraints into holonomic and nonholonomic according to whether the imposed constraints are integrable or not.

The constraints

$$\sum_{k=1}^{N} a_{\alpha k}(x)\dot{x}^k = 0, \quad \alpha = 1, 2, ..., M < N$$

is said to be *integrable* if on $\mathcal{Q}$ there is a smooth (N-M)-dimensional foliation whose leaves are tangent at all their points to the planes defined by the above equations , or what is the same, let

$$\omega_\alpha = \sum_{k=1}^{N} a_{\alpha k}(x) dx^k, \quad \alpha = 1, 2, .., M,$$

be 1-forms on $\mathcal{Q}$; the given constraints are integrable if and only if the 2-forms $d\omega_\alpha, \quad \alpha = 1, 2, .., M$,vanish on the space of admissible (i.e., allowed by the constraints) velocities.

Example 1.1

Suppose that a ball, dynamically symmetric about its center, is placed inside a dynamically symmetric immobile hollow sphere with radius R. The ball roll in the hollow sphere without sliding.

We introduced a fixed system of coordinates axes $Oxyz$ with the origin in the center of the sphere. The velocity of the point of contact of the hollow sphere with the ball can be calculated as follow [Dob]:

$$\vec{v} = \vec{i}(\dot{x} + a(qz - ry) + \vec{j}(\dot{y} + a(rx - pz)) + \vec{k}(\dot{z} + a(py - qx))$$

where the quantities p, q, r are related to the Eulerian angles by the well known kinematic formulas

$$\begin{cases} p = \cos\varphi\dot{\theta} + \sin\varphi\sin\theta\dot{\psi} \\ q = -\sin\varphi\dot{\theta} + \sin\theta\cos\varphi\dot{\psi} \\ r = \cos\theta\dot{\psi} + \dot{\varphi} \end{cases}$$

The condition that the ball roll without sliding the constraints $\vec{v} = 0$ is imposed instantaneously. They express the vanishing of the velocity of the point of contact of the ball with the hollow sphere. In accordance with the above formula the following relations take place

$$\begin{cases} \dot{x} + a(qz - ry) = 0 \\ \dot{y} + a(rx - pz) = 0 \\ \dot{z} + a(py - qx) = 0 \end{cases}$$

By introducing the 1-form

$$\begin{aligned} \omega_1 &= dx + a(\varpi_2 z - \varpi_3 y) \\ \omega_2 &= dy + a(\varpi_3 x - \varpi_1 z) \\ \omega_3 &= dz + a(\varpi_1 y - \varpi_2 x) \end{aligned}$$

i. e.,

$$\varpi_1 = pdt, \quad \varpi_2 = qdt, \quad \varpi_3 = rdt$$

we obtain

$$\begin{cases} d\omega_1 = a(d\varpi_2 - \varpi_1 \wedge \varpi_3)z - a(d\varpi_3 - \varpi_2 \wedge \varpi_1)y \\ d\omega_2 = a(d\varpi_3 - \varpi_1 \wedge \varpi_3)x - a(d\varpi_1 - \varpi_3 \wedge \varpi_2)z \\ d\omega_3 = a(d\varpi_1 - \varpi_3 \wedge \varpi_2)y - a(d\varpi_2 - \varpi_1 \wedge \varpi_3)x \end{cases}$$

In view of the relations

$$d\varpi_1 = \varpi_3 \wedge \varpi_2, \quad d\varpi_2 = \varpi_1 \wedge \varpi_3, \quad d\varpi_3 = \varpi_2 \wedge \varpi_1$$

we deduced that the given constraints by definition are integrable.

The equation of motion of constrained Lagrangian systems can be obtained by using the Lagrange-d'Alembert principle, and they are called the Lagrange's equation with multipliers which are closed systems:

$$(1.2) \qquad \begin{cases} E_j(L) \equiv \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}^j} - \dfrac{\partial L}{\partial x^j} = \displaystyle\sum_{\beta=1}^{M} \mu_\beta \dfrac{\partial f_\beta}{\partial \dot{x}^j}, \quad j = 1, 2, .., N \\ f_\beta(x, \dot{x}) = 0, \quad \beta = 1, 2, .., M \end{cases}$$

If the matrix

$$\Sigma = ((\partial f_\alpha G^{-1}\partial f_\beta)_{\alpha,\beta=1,..,M}),$$

where

$$G = ((\frac{\partial^2 L}{\partial \dot{x}^k \partial \dot{x}^j})_{k,j=1,2,..,N}),$$

is nonsingular then the multipliers can be expressed as function of the state of the system. In this case the equations (1.2) are differential equations on $\mathcal{D}$.

Example 1.2

Consider a skate on an inclined plane Ξ with Cartesian coordinates x, y; we shall assume that the y-axis is horizontal, while the x-axis is directed downward. Let (x, y)

be the coordinates of the point of contact of the "balanced" skate with the plane, and let φ be the rotation angle, measured from the x-axis. The equation of the non-integrable constraint is

$$\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0.$$

For appropriate choice of mass, length, and time units, we can write the Lagrangian as

$$L = \frac{1}{2}((\dot{x})^2 + (\dot{y})^2 + (\dot{\varphi})^2) + gx.$$

The corresponding Lagrange equation with multipliers are the following

$$\begin{cases} \ddot{\varphi} = 0 \\ \ddot{x} = g + \mu\sin\varphi \\ \ddot{y} = \mu\cos\varphi \end{cases}$$

These equations are readily integrated. In fact after straightforward calculations we obtain that, if for example, at the initial time

$$\begin{cases} \varphi|_{t=0} = 0, & \dot{\varphi}|_{t=0} = \omega \\ x|_{t=0} = 0, & \dot{x}|_{t=0} = \dot{x}_0 \\ y|_{t=0} = 0, & \dot{y}|_{t=0} = 0 \end{cases} \tag{1.3}$$

we deduced the relations

$$\begin{cases} \dot{\varphi} = \omega \\ \dot{x} = \dfrac{g}{\omega}\sin\varphi\cos\varphi + \dot{x}_0\cos\varphi \\ \dot{y} = \dfrac{g}{\omega}\sin^2\varphi + \dot{x}_0\sin\varphi \end{cases}$$

if $\omega \neq 0$, Clearly, in this case the nonholonomic skate moves sideways along the cycloid

$$\begin{cases} x = \dfrac{g}{2\omega^2}\sin^2(\omega\, t) + \dot{x}_0\sin\omega\, t \\ y = \dfrac{2g}{4\omega^2}(2\omega\, t - \sin(2\omega\, t)) - \dot{x}_0\cos\omega\, t \end{cases}$$

and, if $\omega = 0$ then the skate moves sideways along the straight lines

$$\begin{cases} x = \dfrac{g\,t^2}{2} + \dot{x}_0\,t \\ y = 0 \end{cases}$$

Clearly, if $g = 0, \quad \omega \neq 0$ then the skate moves sideways the circumferences

$$\begin{cases} x = \dot{x}_0 \sin\omega\,t \\ y = -\dot{x}_0 \cos\omega\,t \end{cases}$$

1.4 VACONOMIC MECHANICS

The d'Alembert-Lagrange principle is not be only rational way of defining the motion of constrained Lagrangian systems. Instead, we may use the Hamilton's principle, whereby the motions of the system are extremal of the variational problem of Lagrange and the equation of motion can be obtained as the Euler-Lagrange equations for an extended singular Lagrangian:

$$\mathcal{L} : TQ \times \mathrm{R}^M \longrightarrow \mathrm{R}$$

$$\mathcal{L} = L - \sum_{\alpha=1}^{M} \lambda_\alpha f_\alpha$$

$$\begin{cases} E_j(\mathcal{L}) = 0, \quad j = 1, 2, ..., N \\ \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\lambda}_\alpha} - \dfrac{\partial \mathcal{L}}{\partial \lambda_\alpha} = f_\alpha \end{cases}$$

or, what is the same,

$$\begin{cases} E_j(L) = \displaystyle\sum_{\beta=1}^{M} (\dot{\lambda}_\beta \frac{\partial f_\beta}{\partial \dot{x}^j} + \lambda_\beta E_j(f_\beta)), \quad j = 1, 2, .., N \\ f_\beta(x, \dot{x}) = 0, \quad \beta = 1, 2, .., M \end{cases} \tag{1.4}$$

The mathematical model of motion of constrained Lagrangian system based on this extension of Hamilton's principle, proposed by V.V.Kozlov [Kozlov2], will be referred to hereafter simply as *vakonomic mechanic.* Clearly, the equations (1.2) differ from the equations of vakonomic mechanic by the presence of the term

$$\sum_{\beta=1}^{M} \lambda_\beta E_j(f_\beta),$$

In the case of the integrable constraints vakonomic mechanics reduces to holonomic mechanics.

The vakonomic model (also called dynamical optimization subject to nonholonomic constraints) have received a lot attention in recent years (see for instance [Arnold et all, Cardin , Cortés, Zampieri]) and is used in is used in mathematical economics (growth economic theory), subriemannian geometry, motion of microorganism, etc..

Example 1.3

We shall illustrate the vaconomic model in the nonholonomic skate described in the example 1.2

The equations of motion (1.4) in this case are the following

$$\begin{cases} \frac{d}{dt}(\dot{x} - \lambda \sin\varphi) = g \\ \frac{d}{dt}(\dot{y} + \lambda \cos\varphi) = 0 \\ \ddot{\varphi} = -\lambda\,(\dot{x}\cos\varphi + \dot{y}\sin\varphi) \end{cases}$$

For the initial conditions (1.3) we obtain that

$$\begin{cases} \dot{x} = \lambda \sin\varphi + g\,t + \dot{x}_0 \\ \dot{y} = -\lambda \cos\varphi + \lambda_0 \\ \ddot{\varphi} = -\lambda\,(\dot{x}\cos\varphi + \dot{y}\sin\varphi) \end{cases}$$

where $\lambda_0 = \lambda|_{t=0}$.

In view of the constraints we deduce that

$$\lambda = -(gt + \dot{x}_0)\sin\varphi + \lambda_0\cos\varphi.$$

Hence, the equations describing the motion of the point of contact in Cartesian coordinates are

$$\begin{cases} \dot{x} = ((gt + \dot{x}_0)\cos^2\varphi - \lambda_0\cos\varphi\sin\varphi \\ \dot{y} = ((gt + \dot{x}_0)\sin\varphi\cos\varphi + \lambda_0\cos^2\varphi \end{cases} \tag{1.5}$$

where φ is a solution of the equation

$$\ddot{\varphi} = ((gt + \dot{x}_0)^2 - \lambda_0^2)\sin\varphi\cos\varphi - \lambda_0(gt + \dot{x}_0)\cos 2\varphi \tag{1.6}$$

In particular if

$$g = 0, \quad \lambda_0 = c\cos\varepsilon, \quad \dot{x}_0 = c\sin\varepsilon$$

after integration we obtain the following equation

$$\dot{\varphi}^2 = \omega^2 + c^2(\sin^2\varepsilon - \sin^2(\varphi + \varepsilon)) \tag{1.7}$$

Hence,

$$\int_0^{\sigma} \frac{d\sigma}{\sqrt{1 - \kappa^2\sin^2\sigma}} = \sqrt{\omega^2 + c^2\sin^2\varepsilon t} + K(\kappa)$$

where

$$K(\kappa) = \int_0^{\frac{\pi}{2}} \frac{d\sigma}{\sqrt{1-\kappa^2\sin^2\sigma}} - \sqrt{\omega^2+c^2\sin^2\varepsilon}\frac{\pi}{2}, \quad \sigma = \varphi+\varepsilon$$

$$\kappa = \frac{|c|}{\sqrt{\omega^2+c^2\sin^2\varepsilon}}.$$

As a consequence

$$\begin{cases} \sin\sigma = sn(\sqrt{\omega^2+c^2\sin^2\varepsilon}t + K(\kappa)), \\ \cos\sigma = cn(\sqrt{\omega^2+c^2\sin^2\varepsilon}t + K(\kappa)) \end{cases}$$

where cn, sn are the Jacobi functions.

After integration (1.5)+(1.7), we deduce the following solution of the given differential equations

$$(1.8) \qquad \begin{cases} x = \int_0^t cn(\rho t + K(\kappa)\Big(\lambda_0 sn(\rho t + K(\kappa)) - \dot{x}_0 cn(\rho t + K(\kappa)\Big)dt \\ y = \int_0^t cn(\rho t + K(\kappa))\Big(\dot{x}_0 sn(\rho t + K(\kappa) + \lambda_0 cn(\rho t + K(\kappa)\Big)dt \end{cases}$$

where $\rho = \omega^2 + c^2\sin^2\varepsilon = \omega^2 + \dot{x}_0^2$.

In particular if

$$\varphi \equiv 0, \quad \varepsilon = \frac{\pi}{2},$$

we obtain the straight line

$$\begin{cases} x = -\dot{x}_0\, t \\ y = 0 \end{cases}$$

It is interesting to compare the motion(1.8) with that of the nonholonomic skate, which for the same initial conditions moves sideways along the circumferences (see example 1.2)

Several authors have discussed the domain of the vakonomic and nonholonomic mechanics (see for instance [Zampieri]). The solutions of the resulting dynamical systems do not coincide, in general, thought there are examples in which the nonholonomic solutions can be seen as solutions of the constrained variational problem. In [Favr]the author obtain conditions in some particular cases for the equivalence between both formulations. In [Cortés et all] the authors present a unified geometrical formulation of dynamics of nonholonomic and vakonomic systems.

The following result we can find in particular in [Arnold et all]:

The principle of determinacy is not valid for vakonomic systems with non-integrable constraints

This result follow in particular from the example 1.3. The existence of the arbitrary parameter λ_0 in (1.8) showed that, in general, for the given initial conditions do not exist the unique solutions of the differential equations in the vaconomic model

2. THE A-MODEL IN MECHANICS

2.1 TRANSPOSITIONAL RELATIONS IN MECHANICS.

One of the most general integral variational principles is the principle of stationary action or Hamiltonian principle, according to which the actual motion of the system is such that [Sulgin, NK1, Rumiansev]

$$\int_{t_0}^{t_1} (\delta T + \delta A) dt = 0 \tag{2.1}$$

where δT is the variation of the kinetic energy and δA is the virtual work of the forces applied to the system.

The expression (2.1), can be simplified if the field of forces have a potential, i.e.,

$$\delta A = \delta U.$$

Introducing the Lagrange function $L = T + U$ the formula (2.1) can be transform to the form:

$$\int_{t_0}^{t_1} \delta L dt = 0 \tag{2.2}$$

From the formal point of view, the principle of stationary action in the above form is equivalent to the problem of variational calculus. However, despite the superficial similarity, they differ essentially. Namely, in mechanics the symbol δ stands for a virtual variation, i.e., not an arbitrary infinitesimally small variation but a displacement compatible with the constraints imposed upon the system. Thus, it is only in the case of the holonomic system, for which the number degrees of freedom is equal to the number of generalized coordinates, that the virtual variations are arbitrary and the principle of stationary action (2.2) is completely equivalent to the corresponding problem in variational calculus. An important differences arises for the system with nonholonomic constraints, when the variations of the generalized coordinates are connected by subsidiary relations (*Chetaev condition*)

$$\begin{cases} f_\alpha(x, \dot{x}, t) = 0 \\ \displaystyle\sum_{k=1}^{N} \frac{\partial f_\alpha}{\partial \dot{x}^k} \delta x^k = 0, \quad \alpha = 1, 2, .., M \end{cases} \tag{2.3}$$

After that Hertz introduce the nonholonomic mechanics arise the question on the extension to this system the results of mechanic of holonomic systems. Hertz [Hertz] was the first which study the problem on the application of the Hamiltonian principle for system with non-integrable constraints.

For nonholonomic system, the curve obtained form the curve of the actual motion by means of an infinitesimally small virtual variation is not, in general, a kinematically possible trajectory. This circumstance prompted the conclusion that the principle of stationary action cannot be applied for nonholonomic system, at least not in the form (2.1) or (2.2), in which it is usually employed for holonomic systems. The essence of the problem of the applicability of this principle for nonholonomic system remained unclarified.

Appel in[Appel] in correspondence with Hertz's ideas affirmed that it is not possible to apply the Hamiltonian principle for system with nonintegrable constraints

In order to clarify the situation, it is sufficient to note that the question of the applicability of the principle of stationary action to nonholonomic system is intimately related to the question of *transpositional relations*

$$\delta\frac{dx^k}{dt} - \frac{d}{dt}\delta x^k, \quad k = 1, 2, ..., N. \tag{2.4}$$

The point is that the derivation of the principle of stationary action in the form (2.1) presupposes that the operation of differentiation with respect to the time $\frac{d}{dt}$ and virtual variation δ commute for all generalized coordinates $x^1, x^2, ..., x^N$.

To see this we write down the d'Alembert -Lagrange principle (1.1) in the form

$$\sum_{k=1}^{N}(E_k(T) - Q_k)\delta x^k = 0$$

where

$$\sum_{k=1}^{n}\frac{\partial \mathbf{r}_k}{\partial x^j}\mathbf{F}_k == Q_j, \quad \xi_k = \sum_{j=1}^{N}\frac{\partial \mathbf{r}_k}{\partial x^j}\delta x^j$$

Performing identity transformation, we obtain

$$\delta T + \sum_{j=1}^{N}Q_j\delta x^j + \sum_{k=1}^{N}\frac{\partial T}{\partial \dot{x}^k}(\frac{d}{dt}\delta x^k - \delta\frac{dx^k}{dt}) = \sum_{k=1}^{N}\frac{d}{dt}(\frac{\partial T}{\partial \dot{x}^k}\delta x^k) \tag{2.5}$$

where

After integration (2.5) under the condition $\delta x(t_1) = \delta x(t_2) = 0$ we deduce

$$\int_{t_0}^{t_1}\big(\delta T + \delta A + \sum_{k=1}^{N}\frac{\partial T}{\partial \dot{x}^k}(\frac{d}{dt}\delta x^k - \delta\frac{dx^k}{dt}))dt = 0, \tag{2.6}$$

This expression may be regarded as the most general mathematical formulation of the principle of stationary action, suitable for the both holonomic and nonholonomic systems. As regards holonomic systems, the relation

$$\delta\frac{dx^k}{dt} - \frac{d}{dt}\delta x^k = 0, \quad k = 1, 2, ..N$$

always holds for them and, hence, the expression (2.6) is identical with (2.1)

In the case of a nonholonomic system, the special form for the principle stationary action deduced from (2.6) will depend on the point of view adopted with regard to the transpositional relations.

What are then correct transpositional relations? Until recently,there has not been a common point of view concerning the commutativity of the operation of differentiation with respect to the time and virtual variation when there are nointegrable kinematic constraints. Two standpoints have been maintained. According to one (supported, for example, by Volterra and Hamel), the operations $\frac{d}{dt}$ and δ commute for all true $x^1, x^2, ..., x^N$, irrespective of whether the system is holonomic or nonholonomic. According to the other point of view (supported by Suslov, Vorones, Levi-Civita and Amaldi) the operations $\frac{d}{dt}$ and δ commute for holonomic systems. In the case of nonholonomic systems, the transpositional relations are equal to zero only for those generalized coordinates whose variations (in accordance with the equations of nonholonomic constraints) are independent. For the remaining coordinates, the transpositional relations are to be derived on the basis of the equations of the nonholonomic constraints, and they are different of zero. The second point of view acquired general acceptance and the first point of view was erroneous [NF].

From the results which we exposed below it is possible to observe that the second point of view in general is correctly only for the so called Chapliguin systems. There exist a lot of example in which the independent variations generated nonzero transpositional relations.

What is meant by the expressions

$$\frac{d}{dt}\delta x^k - \delta\frac{dx^k}{dt}?$$

The answer to this question we can find in particular in [NF2, Kirguetov] .

The different form of definition the transpositional relations leading to the new form of the equations of motion. As an example, let us consider one such case which we describe below, which seems is the most general form of determination theses relations. .

We shall applied the Hamiltonian principle in form (2.6) for an arbitrary Lagrangian system $(\mathcal{Q}, L)$, i.e.,we shall consider that

$$\int_{t_0}^{t_1}(\delta L + \sum_{k=1}^{N}\frac{\partial L}{\partial \dot{x}^k}(\delta\frac{dx^k}{dt} - \frac{d}{dt}\delta x^k))dt = 0, \tag{2.7}$$

The aim of this section is to deduced from this principle a mathematical model to describe the behavior of mechanical system of the most general nature which is possible.

2.2 THE A-MODEL.

We apply the Hamiltonian principle (2.7) for the case when the Lagrangian function L is a smooth function on $T\mathcal{Q} \times \mathrm{R}$:

$$L = L_0(x, \dot{x}, t) - \sum_{j=1}^{N}\lambda_j(t)L_j(x, \dot{x}, t) \tag{2.8}$$

where $L_j, \quad j=0,1,..,M$ are smooth functions which satisfies the following conditions

1.

$$det(\frac{\partial^2 L_0}{\partial \dot{x}^k \partial \dot{x}^j}) \neq 0$$

2. The 1-form

$$\Theta_\alpha = \sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}^k}(dx^k - \dot{x}^k dt) + L_\alpha\, dt, \quad \alpha = 1,2,..,N$$

are independent, i.e., the matrix

$$W = \begin{pmatrix} \Theta_1(\partial_1) & \Theta_1(\partial_2) & \dots & \Theta_1(\partial_N) \\ \Theta_2(\partial_1) & \Theta_2(\partial_2) & \dots & \Theta_2(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ \Theta_S(\partial_1) & \Theta_S(\partial_2) & \dots & \Theta_S(\partial_N) \\ \Theta_{S+1}(\partial_1) & \Theta_{S+1}(\partial_2) & \dots & \Theta_{S+1}(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ \Theta_N(\partial_1) & \Theta_N(\partial_2) & \dots & \Theta_N(\partial_N) \end{pmatrix}$$

is non-degenerate, where $\partial_k = \frac{\partial}{\partial x^k}$.

3. The functions $\lambda_j : I \subseteq \mathrm{R} \to \mathrm{R}, \quad j = 1,2,..N$ are function class $C^r(I)$, $r \geq 1$, (Lagrangian multipliers) which we choose in such a way that

$$\dot{\lambda}_j(t)\, L_j(x, \dot{x}, t) = 0, \quad j = 1,2,..,N \tag{2.9}$$

With respect to the variation δx^j we suppose that they generated the transpositional relations:

$$\delta \frac{dx^j}{dt} - \frac{d}{dt}\delta x^j = \sum_{k=1}^{N} {(A_k^j)}^T(t, x, \dot{x}, \ddot{x})\delta x^k\,, j = 1,2,..N \tag{2.10}$$

where $A = (A_k^j))$ is a matrix which we determine below, and A^T is the transposition of the matrix A.

Proposition 2.1

Let us suppose that the functions

$$\delta x^k(t), \quad k = 1,2,..,N$$

are arbitrary functions on the interior of the interval $[t_0,\, t_1]$ and are vanishing at the end of the interval, i.e.,

$$\delta x^j(t_0) = \delta x^j(t_1) = 0, \quad j = 1,2,..,N$$

Then (2.6) holds if and only if the path $\gamma(t) = (x^1(t), x^2(t), ..., x^N(t))$ is a solution of the differential equations of the second order

$$D_\nu(L) \equiv E_\nu(L) - \sum_{j=1}^{N} A_\nu^j \frac{\partial L}{\partial \dot{x}^j} = 0,\, \nu = 1,..N \tag{2.11}$$

or, what is the same,

$$D_\nu(L_0) - \sum_{\alpha=1}^{M}(\lambda_\alpha D_\nu(L_\alpha) + \dot{\lambda}_\alpha \frac{\partial L}{\partial \dot{x}^\nu}) = 0, \quad \nu = 1, 2, .., N \tag{2.12}$$

The proof we obtain from (2.7)+(2.8)+(2.10).

In the all study below cases we shall determine the matrix A in such a way that

a)

$$\begin{cases} E_k(L_0) = \sum_{j=1}^{N} A_k^j \frac{\partial L_0}{\partial \dot{x}^j} + \sum_{\alpha=1}^{M} \dot{\lambda}_\alpha \frac{\partial L_\alpha}{\partial \dot{x}^k}) \\ E_k(L_\alpha) = \sum_{j=1}^{N} A_k^j \frac{\partial L_\alpha}{\partial \dot{x}^j}, \quad \alpha = 1, 2, .., M \end{cases} \tag{2.13}$$

or,

b)

$$\begin{cases} E_k(L_0) = \sum_{\alpha=1}^{M} \dot{\lambda}_\alpha \frac{\partial L_\alpha}{\partial \dot{x}^k}, \\ \sum_{j=1}^{N} A_k^j \frac{\partial L_0}{\partial \dot{x}^j} = E_k(\frac{df_N(x)}{dt}) = 0, \\ E_k(L_\alpha) = \sum_{j=1}^{N} A_k^j \frac{\partial L_\alpha}{\partial \dot{x}^j}, \\ \alpha = 1, 2, .., M, \quad k = 1, 2, .., N \end{cases} \tag{2.14}$$

Clearly, if the relations a) or b) hold then

$$\delta L_\alpha = \frac{d}{dt}(\frac{\partial L_\alpha}{\partial \dot{x}^k}\delta x^k)$$

or, what is the same,

$$\delta\Theta_\alpha(\frac{d}{dt}) = \frac{d}{dt}\Theta_\alpha(\delta)$$

where δ and $\frac{d}{dt}$ are the variation and velocity vector field:

$$\begin{cases} \delta = \sum_{k=1}^{N} \delta x^k \frac{\partial}{\partial x^k} \\ \frac{d}{dt} = \sum_{k=1}^{N} \dot{x}^k \frac{\partial}{\partial x^k} + \frac{\partial}{\partial t} \end{cases}$$

Corollary 2.1
Let $E(L)$ be a matrix:

$$E(L) = \begin{pmatrix} E_1(L_1) & E_2(L_1) & \dots & E_N(L_1) \\ E_1(L_2) & E_2(L_2) & \dots & E_N(L_2) \\ \vdots & \vdots & \dots & \vdots \\ E_1(L_S) & E_2(L_S) & \dots & E_N(L_S) \\ \vdots & \vdots & \dots & \vdots \\ E_1(L_{N-1}) & E_2(L_{N-1}) & \dots & E_N(L_{N-1}) \\ E_1(\Upsilon) & E_2(\Upsilon) & \dots & E_N(\Upsilon) \end{pmatrix} \tag{2.15}$$

where Υ is a function which we choose as follow: $\Upsilon = L_N$ in the case a) or $\Upsilon = \frac{df_N(x)}{dt}$ in the case b).

Then the equations (2.13) and (2.14) admit the following matrix representation

$$WA^T = E(L) \tag{2.16}$$

Let $\vec{E}(L_0)$, $\vec{\partial}L_0$, $\vec{\dot{\lambda}}$ are the following vectors

$$\vec{E}(L_0) = col(E_1(L_0),\, E_2(L_0), \dots, E_N(L_0)$$
$$\vec{\partial}(L_0) = col(\frac{\partial L_0}{\partial \dot{x}^1},\, \frac{\partial L_0}{\partial \dot{x}^2}, \dots, \frac{\partial L_0}{\partial \dot{x}^N})$$
$$\vec{\dot{\lambda}} = col(\dot{\lambda}_1,\, \dot{\lambda}_2, \dots \dot{\lambda}_N)$$

By solving (2.16) with respect to the matrix A we prove the following proposition

Proposition 2.2

The differential equations (2.12) under the restrictions (2.16) can be represented in the following matrix form :

$$\vec{E}(L_0) = (W^{-1}E(L))^T \vec{\partial}(L_0) + W^T \vec{\dot{\lambda}} \tag{2.17}$$

Corollary 2.2
The transpositional relations in the A-model admit the following representation

$$\delta \frac{\vec{dx}}{dt} - \frac{d}{dt}\vec{\delta}x = W^{-1}E(L)\vec{\delta}x \tag{2.18}$$

Definition 2.1

The mathematical model basis on the equations (2.17), transpositional relations (2.18) and conditions (2.9) we shall called the *A-model.*

In the local coordinates x on $\mathcal{Q}$, the equations (2.17) can be written in the explicit form:

$$\sum_{k=1}^{N} G_{kj}(x,\dot{x},t)\ddot{x}^k + \Psi_j(x,\dot{x},t) = 0, \quad j = 1,2,...,N \tag{2.19}$$

where $G = (G_{kj})$ is the matrix:

$$G_{kj} = \frac{\partial^2 L_0}{\partial \dot{x}^k \partial \dot{x}^j} - \sum_{s=1}^{N} \frac{\partial A_j^s}{\partial \ddot{x}^k} \frac{\partial L_0}{\partial \dot{x}^s} \tag{2.20}$$

Corollary 2.3

The principle of determinacy is valid for the mechanical systems in the A-model if and only if the matrix G is non-degenerate matrix.

In fact, if $detG \neq 0$ then the equations (2.19) can be resolved with respect to "accelerations." This implies, in particular, that the state of the system at time $t_0 \in \Delta$ uniquely determines its motion.

Corollary 2.4

In the A-model the dependent variations always conduce to the nonzero transpositional relations. With respect to transpositional relations correspond to independent variations its must be nonzero.

3. VALIDITY OF THE A-MATHEMATICAL MODEL IN MECHANICS

In this chapter, we shall, comparing with the well known mathematical model in mechanics, show the importance of the proposed above mathematical model.

Formally the relations with the vaconomic mechanics is easy to obtain. In fact, if in the A-model we require that the all variations of the coordinate produce the zero transpositional relations then we obtain the vaconomic model.

3.1 LAGRANGIAN CASE

The Lagrangian equations which describe the behavior of holonomic system we can obtained from the A-model by supposing that

$$\dot{\lambda}_\alpha = 0, \quad L_\alpha = \frac{df_\alpha(x)}{dt}, \quad \alpha = 1,2,..,N$$

where f_1, f_2, ..., f_N are independent smooth functions.Hence the 1-forms Θ_α :

$$\Theta_\alpha = df_\alpha$$

The matrix W for the Lagrangian case is the following

$$W = \begin{pmatrix} df_1(\partial_1) & df_1(\partial_2) & \dots & df_1(\partial_N) \\ df_2(\partial_1) & df_2(\partial_2) & \dots & df_2(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ df_S(\partial_1) & df_S(\partial_2) & \dots & df_S(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ df_N(\partial_1) & df_N(\partial_2) & \dots & df_N(\partial_N) \end{pmatrix} \tag{3.1}$$

Clearly, the matrix $E(L)$ is a zero matrix, hence the matrix A is a zero matrix .

As a consequence the propose model coincide with the Lagrangian model and the equation of motion are:

$$E_k(L_0) = 0, \quad k = 1, 2, .., N$$

The transpositional relations (2.18) in this case take the form

$$\delta\dot{x}^\alpha - \frac{d\delta x^\alpha}{dt} = 0, \quad \alpha = 1, 2, ..N$$

which is well known relations in the mechanic of holonomic systems.

Example 3.1

In this example we study special Lagrangian system and established the relations with the so called Cartesian approach.

In the history of mechanics, there have been two point of views for studying mechanical systems: The Newtonian and the Cartesian. In "Philosophiae Naturalis Principia Mathematica" (1687),Newton considers that movements of celestial bodies can be described by differential equations of the second order. To determine their trajectory, it is necessary to give the initial position and velocity. Descartes in "Principia Philosophiae" (1644) proposed that the behavior of the celestial bodies be studied from another point of view. He considered, that bodies are moved by ether. As a consequence, the equation of motion must be of the first order

$$\dot{\mathbf{x}} = \mathbf{v}(x)$$

Hence, to determine the trajectory from Descartes's point of view it is necessary to give only the initial position. Decartes gave no principles for constructing the field $\mathbf{v}$ for different mechanical systems. As Kozlov affirm [Kozlov3] "solving dynamics problem is possible inside the configuration space".

A main achievement of Newton was perceiving that the dynamics of real systems are described by second-order differential equations. To deduce the equations of motion to the investigation of a dynamics systems (i.e., to first order equation), it is necessary to double the dimension of the position space and to introduce the auxiliary phase space. However, we are interested not in the phase trajectories themselves but in their projection on the configuration space.

Now we shall introduce the following notation and concept.

Let $\mathcal{Q}$ be a smooth manifold of the dimension N with local coordinates $x = (x^1, ..., x^N)$ and equipped by the Riemann metric $G = (G_{kj}(x))$.

By $\xi(\mathcal{Q})$, $\Lambda(\mathcal{Q})$, ∇ we denote respectively the Lie algebra of vector fields on $\mathcal{Q}$ and the algebra of the 1-form on $\mathcal{Q}$, and the connection:

$$\begin{aligned} \nabla :&\xi(\mathcal{Q}) \times \xi(\mathcal{Q}) \longmapsto \xi(\mathcal{Q}) \\ &(u, v) \longmapsto \nabla_u v \end{aligned}$$

which is R lineal with respect to v and C^∞ lineal with respect to v and is compatible with metric G, i.e., $\nabla_u G(v, w) = 0, \quad \forall u, v, w \in \xi(\mathcal{Q})$.

We shall study a Lagrangian mechanical system with N degree of freedom with Lagrangian function L_0 :

$$L_0 = \frac{1}{2}||\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})||^2 = \frac{1}{2}\sum_{j,k=1}^{N} G_{jk}(x)(\dot{x}^j - v^j)(\dot{x}^k - v^k)$$

where $\mathbf{v}$ is a vector field:

$$\mathbf{v} = det\begin{pmatrix} \Omega_1(\partial_1) & \Omega_1(\partial_2) & \dots & \Omega_1(\partial_N) & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_M(\partial_1) & \Omega_M(\partial_2) & \dots & \Omega_M(\partial_N) & 0 \\ \Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \dots & \Omega_{M+1}(\partial_N) & \lambda_{M+1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_N(\partial_1) & \Omega_N(\partial_2) & \dots & \Omega_N(\partial_N) & \lambda_N \\ \partial_1 & \partial_2 & \dots & \partial_N & 0 \end{pmatrix},$$

where $\partial_k = \frac{\partial}{\partial x^k}$, we shall consider that

$$\Omega_1, \Omega_2, ...\Omega_M \in \Lambda(\mathcal{Q}), \quad M \leq N - 1$$

are given 1-forms, and $\Omega_{M+1}, \Omega_{M+2}, ...\Omega_N$, are arbitrary 1-forms on $\mathcal{Q}$. Furthermore, we assume that they are pointwise independent i.e.

$$\Upsilon \equiv \Omega_1 \wedge \Omega_2 ... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N) \neq 0,$$

The functions $\lambda_j, \quad j = M+1, ..., N$ are arbitrary functions on $\mathcal{Q}$

The vector field $\mathbf{v}$ has the following properties

1.
$$\begin{aligned} \Omega_j(\mathbf{v}) &= 0, \quad j = 1, 2, .., M \\ \Omega_j(\mathbf{v}) &= -\Upsilon\lambda_j, \quad j = M+1, .., N. \end{aligned}$$

2. We deduce that the vector $\mathbf{v}(x) = (v^1(x), ..., v^N(x))^T$ can be represented as follows

$$\mathbf{v}(x) = \mathcal{M}^{-1}\lambda,$$

where $\mathcal{M} = \Big(\Omega_j(\partial_k)_{j,k=1,..N}\Big), \quad \lambda = -\Upsilon\,(0,..,0,\lambda_{M+1},..\lambda_N).$

3. Let σ be the 1-form associated with the vector field $\mathbf{v}$, i.e.,

$$\sigma = (\mathbf{v}(x), dx) \equiv \sum_{j,k=1}^{N} G_{jk}(x)v^j(x)dx^k \equiv \sum_{k=1}^{N} v_k dx^k$$

then the 2-form $d\sigma$:

$$d\sigma = \frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(x)\Omega_j \wedge \Omega_k,$$

where $A = (a_{jk})$ is a matrix such that

$$a_{jk} = (-1)^{j+k-1}\frac{1}{\Upsilon}d\sigma \wedge \Omega_1 \wedge ... \wedge \widehat{\Omega}_k .... \wedge \widehat{\Omega}_j .... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N)$$

$\widehat{\Omega}_j$, $\widehat{\Omega}_k$ means that these elements are omitted.

It is clear that the contraction of $d\sigma$ along $\mathbf{v}$ is

$$\iota_{\mathbf{v}} d\sigma = \sum_{j=1}^{N} \Lambda_j \Omega_j, \quad \text{where}\, \Lambda \equiv col(\Lambda_1, \Lambda_2, ..., \Lambda_N) = A^T\lambda.$$

We shall analyze the differential equations

$$\dot{\mathbf{x}} = \mathcal{M}^{-1}\lambda \tag{3.2}$$

under the conditions

$$\begin{cases} \Lambda_j = 0, \quad j = M+1,..,N \\ \Upsilon = \Omega_1 \wedge \Omega_2 .... \wedge \Omega_N(\partial_1,\, \partial_2,\, ...,\partial_N) \neq 0 \end{cases} \tag{3.3}$$

Proposition 3.1 *The differential equations*

$$\dot{\mathbf{x}} = \mathbf{v}(x), \quad x \in \mathcal{Q}$$

are invariant relationship of the Lagrangian equations with Lagrangian function

$$L_0 = \frac{1}{2}||\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})||^2$$

In fact, by derivation we deduce that $\nabla_{\dot{\mathbf{x}}}(\dot{\mathbf{x}} - \mathbf{v}(x)) = 0$, or,

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{\mathbf{x}}} L_0) = 0,$$

which are equivalent to Lagrangian equations with the Lagrangian function L_0 given above. It is easy to show that these equations admits the representation

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{\mathbf{x}}^j} T) = \omega(\partial_j) + \nabla_{\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})} v_j$$

where

$$T = \frac{1}{2}||\dot{\mathbf{x}}||^2,$$
$$\omega = d\frac{||\mathbf{v}||^2}{2} + \iota_{\mathbf{v}} d\sigma,$$

σ is the 1- form associated with the vector field $\mathbf{v}$.

We shall study the case when (3.2) and (3.3) hold. The differential equations which describe the behavior of such mechanical systems under these restrictions can be represented as follows

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{\mathbf{x}}^k} T) = \frac{\partial \frac{1}{2}||\mathbf{v}||^2}{\partial x^k} + \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_k). \tag{3.4}$$

These differential equations can be interpreted as the equations of motion of nonholonomic mechanical systems with an active potential field of force with potential U :

$$U = \frac{1}{2}||\mathbf{v}(x)||^2 + U_0, \quad U_0 = const.$$

and with the reactive forces which have the components

$$(\sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_1), \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_2), ..., \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_N)),$$

generated by the constraints $\sum_{k=1}^{N} \Omega_j(\partial_k)\dot{x}^k = 0$.

The studying of the behavior of the nonholonomic systems by using the equations (3.2)+(3.3) or (3.4) we called Cartesian and Lagrangian approach respectively [Sad, Ram].

In particular, for a constrained particle in R^3 the Cartesian and Lagrangian approach can be obtained as follow.

The Lagrangian equation of motion of the particle in this case can be rewritten in the following form:

$$\ddot{\mathbf{r}} = \omega(\vec{\partial})$$

where

$$\mathbf{r} = col(x, y, z) \quad \vec{\partial} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\frac{\partial}{\partial z})$$

and ω is a 1-form:

$$\omega = d(\frac{1}{2}||\mathbf{v}||^2) + \sum_{j=1}^{3} \Lambda_j \Omega_j + (([\dot{\mathbf{r}} - \mathbf{v}) \times rot\mathbf{v}], d\vec{\mathbf{r}})$$

where

$$rot\mathbf{v} = (\partial_y \mathbf{v}(z) - \partial_z \mathbf{v}(y))\partial_x + (\partial_z \mathbf{v}(x) - \partial_x \mathbf{v}(z))\partial_y + (\partial_x \mathbf{v}(y) - \partial_y \mathbf{v}(x))\partial_z,$$

and

$$\begin{cases} \Lambda_1 = \lambda_2\Omega_3(rot\mathbf{v}) - \lambda_3\Omega_2(rot\mathbf{v}) \\ \Lambda_2 = \lambda_3\Omega_1(rot\mathbf{v}) \\ \Lambda_3 = -\lambda_2\Omega_1(rot\mathbf{v}) \end{cases}$$

In this case we have that (3.2)+(3.3):

$$\begin{cases} \dot{\mathbf{r}} = \mathbf{v} \\ \Omega_1(rot\mathbf{v}) = 0 \end{cases}.$$

The equations (3.4) take the form:

$$\ddot{\mathbf{r}} = \vec{\partial}\frac{1}{2}||\mathbf{v}||^2 + \Lambda_1\Omega_1(\vec{\partial})$$

which can be interpreted as equations of motion of a constrained particle in R^3 with the constraint

$$\Omega_1(\dot{\mathbf{r}}) = 0$$

We shall illustrate the above ideas in the following particular case.

Let us suppose that

$$\begin{cases} \Omega_1 = dx + a(x,y,z)dy, \\ \Omega_2 = dy, \\ \Omega_3 = dz \end{cases}$$

then

$$\begin{cases} \mathbf{v} = a\lambda_2\partial_x - \lambda_2\partial_y - \lambda_3\partial_z \\ \Omega_1(rot\mathbf{v}) = 0 \Leftrightarrow \partial_z((1+a^2)\lambda_2^2) + (a\partial_x\lambda_3 - \partial_y\lambda_3)\lambda_2 = 0 \\ \Lambda_1 = -\lambda_2(\partial_y(a\lambda_2) + \partial_x\lambda_2) - \lambda_3(\partial_x\lambda_3 + \partial_z(a\lambda_2)) \end{cases}$$

We shall study the case when $a = a(z)$, $\lambda_2 = \frac{A}{\sqrt{a^2+1}}$, $\lambda_3 = b_2(z)$, where A is an arbitrary constant, b_2 is an arbitrary function.

The equations generated by the vector field $\mathbf{v}$ in this case are

$$\begin{cases} \dot{x} = \dfrac{a(z)A}{\sqrt{1+a^2(z)}} \\ \dot{y} = -\dfrac{A}{\sqrt{1+a^2(z)}} \\ \dot{z} = -b_2(z) \end{cases} \tag{3.5}$$

Corollary 3.1 All the trajectories of the equation of motion of the constrained Lagrangian system

$$< \mathrm{R}^3, L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(z), \mathcal{D} = \{\dot{x} + a(z)\dot{y} = 0\} >$$

coincide with the solutions of (3.5) [Sad].

3.2 LAGRANGIAN CONSTRAINED CASE

We shall established the relations between the equations (2.17) and (1.2). So we shall suppose that the functions $L_\alpha, \quad \alpha = 1, 2, .., M$ are constraints:

$$\text{(3.6)} \qquad \begin{cases} rang(\dfrac{\partial L_\alpha}{\partial \dot{x}^k}) = M < N \\ L_\alpha(x, \dot{x}, t) = 0, \quad \alpha = 1, 2, ..., M \end{cases}$$

From the condition (2.9) we deduced that

$$\dot{\lambda}_j = 0, \quad j = M+1, M+2, ..., N$$

Definition 3.1

The constraints (3.6) are called *ideal constraints* if satisfies the *Chetaev conditions*

$$\Theta_j(\delta) = 0 \Longleftrightarrow \sum_{j=1}^{N} \frac{\partial L_\nu}{\partial \dot{x}^j}\delta x^j = 0, \quad \alpha = 1, 2, .., N$$

In this paper we study only ideal constrains.

To prove that the equations (1.2) coincide with (2.17) in this case we must to determine the matrix A in such a way that the following relations hold

$$\begin{cases} E_k(L_\nu) = \displaystyle\sum_{j=1}^{N} A_k^j \frac{\partial L_\nu}{\partial \dot{x}^j}, \quad k, \nu = 1, 2, ..., S = N-1 \\ \displaystyle\sum_{j=1}^{N} A_k^j \frac{\partial L_0}{\partial \dot{x}^j} = 0, \quad k = 1, 2, ..., N, \end{cases}$$

We firstly shall study the case of integrable constraints, i.e., we suppose that

$$L_\alpha \equiv \frac{df\alpha(x)}{dt} = 0,$$
$$\dot{\lambda}_j = 0, \quad j = M+1, .., N, \alpha = 1, 2, .., M$$

The matrix $E(L)$, is a zero matrix, as a consequence the matrix A is zero matrix.

Clearly that the equations (2.17) in this case takes the form

$$\text{(3.7)} \qquad E_k(L_0) = \sum_{\alpha=1}^{M} \dot{\lambda}_\alpha \frac{\partial f_\alpha}{\partial x^k}, \quad k = 1, 2, ..N$$

and the transpositional relations are all equal to zero, i.e.,

$$\delta\frac{dx^k}{dt} - \frac{d\delta x^k}{dt} = 0, \quad k = 1, 2, ..N$$

Now we shall study the general case when M of the constraints are nonintegrable. By introducing the matrix W and $E(L)$:

$$W = \begin{pmatrix} \Theta_1(\partial_1) & \Theta_1(\partial_2) & \dots & \Theta_1(\partial_N) \\ \Theta_2(\partial_1) & \Theta_2(\partial_2) & \dots & \Theta_2(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ \Theta_M(\partial_1) & \Theta_M(\partial_2) & \dots & \Theta_M(\partial_N) \\ df_{M+1}(\partial_1) & df_{M+1}(\partial_2) & \dots & df_{M+1}(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ df_{N-1}(\partial_1) & df_{N-1}(\partial_2) & \dots & df_{N-1}(\partial_N) \\ \Theta_0(\partial_1) & \Theta_0(\partial_2) & \dots & \Theta_0(\partial_N) \end{pmatrix} \tag{3.8}$$

where $\theta_0 = \sum_{j=1}^N \frac{\partial L_0}{\partial \dot{x}^j}(dx^j - \dot{x}^j dt) + L_0 dt$, and $f_{M+1}, .., f_{N-1}$ are arbitrary independent smooth functions. The matrix $E(L)$ we determine as follow:

$$E(L) = \begin{pmatrix} E_1(L_2) & E_2(L_2) & \dots & E_N(L_2) \\ \vdots & \vdots & \dots & \vdots \\ E_1(L_M) & E_2(L_M) & \dots & E_N(L_M) \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

The equations (2.17) under theses conditions coincide with the Lagrangian equations obtained from the Lagrange-d'Alembert principle with multipliers

$$\dot{\lambda}_\nu = \mu_\nu$$

and are such that

$$E_k(L_0) = \sum_{\nu=1}^M \dot{\lambda}_\nu \frac{\partial L_\nu}{\partial \dot{x}^k} \tag{3.9}$$

The transpositional relations (2.18) for the constrained Lagrangian system take the form

$$\delta \frac{d\vec{x}}{dt} - \frac{d}{dt}\delta\vec{x} = W^{-1}E(L)\delta\vec{x} \tag{3.10}$$

As we observe above, the according to the Suslov, Levi-Civita and Amaldi the operations $\frac{d}{dt}$ and δ, in the case of nonholonomic systems, the relation

$$\delta \frac{dx^j}{dt} - \frac{d\delta\, x^j}{dt} = 0 \tag{3.11}$$

only for those generalized coordinates whose variations (in accordance with the equations of the nonholonomic constraints) are independent.

From (3.10) and more concretely, for the examples given below, we deduced that in A-model, in general, (3.11) can not take place inclusive for the variation which are independent.

Example 3.2

We shall illustrate this case in the constrained particle in R^N with L_0 :

$$L_0 = \frac{1}{2}\sum_{j=1}^{N} \dot{x_j}^2 + U(x)$$

and with the constraints

$$f_1(x) \equiv \sum_{j=1}^{N} x_j^2 - 1 = 0$$

The equations (3.5) can be reexpressed as follows

$$\ddot{x}_k = -\partial_{x^k} U(x) + \dot{\lambda} x_k, \quad k = 1, 2, ..N$$

or, what is the same,

$$\text{(3.12)} \qquad \begin{cases} \dot{x}_k = X_k + \lambda x_k \\ \dot{X}_k = \partial_{x^k} U(x) + \lambda(X_k + \lambda x_k) \end{cases}$$

From the first equations we obtain that

$$\text{(3.13)} \qquad \lambda = -\frac{(x, X)}{(x, x)}$$

By inserting (3.13) into (3.12) we easily deduced the following expression for (3.12) :

$$\begin{cases} \dot{x} = V(X)\, x \\ \dot{V}(X) = V(gradU) \\ (x, x) = 1, \end{cases}$$

where $V(X)$ is a matrix:

$$V(X) = (x_k X_j - x_j X_k)|_{k,j=1,2,\ldots,N}$$

In particular for $N = 3$ the deduced equations can be reexpressed as follow

$$\begin{cases} \dot{x} = [x,\, \tau] \\ \dot{\tau} = [x,\, gradU] \end{cases}$$

where [,] is the standard vector product in R^3, $x = col(x, y, z)$ and $\tau = [x,\, \dot{x}]$

We shall study the particular case when

$$U(x) = (x, Ix) + (a, x) \equiv \sum_{j=1}^{N} (I_j x_j^2 + a_j x_j)$$

where $I_j,\, a_j, \quad j = 1, 2, .., N$ are real constant.

Corollary 3.2

Let U be a function:

$$U = (x, Ix), \quad I = diag(I_1, I_2, ..., I_N) \tag{3.14}$$

Then the N functions

$$F_\nu(x, X) = x_\nu^2 + \sum_{j=1,\, j\neq\nu}^{N} \frac{V_{j\nu}^2}{I_\nu - I_j}, \quad \nu = 1, 2, .., N$$

are first integral of the differential systems (3.12)+(3.14).

The proof it is easy to obtain by considering that

$$\dot{V_{j\nu}} = (I_\nu - I_j) x_j x_\nu, \quad j, \nu = 1, 2, ..., N$$

Hence

$$\dot{F_\nu}(x, X) = x_\nu (\dot{x}_\nu - \sum_{j=1}^{N} V_{j\nu} x^j) \equiv 0$$

Is well known that theses N first integrals are an involution [Moser].

Hence we deduced the completely integrability of this mechanical system.

This example represents an anisotropic N freedom harmonic oscillator constrained to a sphere S^{N-1} in R^N. It was first treated by Neumann (1859) for $N = 3$; later Uhlenbeck (1982) showed it to be completely integrable in the sense of Liouville for arbitrary finite N.

Corollary 3.3

Let us suppose that

$$U = (a, x), \quad a = (a_1, a_2, ..., a_N)$$

then the system (3.12) can be represented as follows [Sad]:

$$\ddot{x}_j = -a_j - (3(a, x) - 2h) x_j$$

where h is an arbitrary constant:

$$\sum_{j=1}^{N} \dot{x}_j^2 = 2h - 2(a, x)$$

By considering that

$$\frac{d^2(a,x)}{dt^2} = -\frac{1}{2}(6((a,x)-\frac{h}{3})^2 + 2(a,a) - \frac{2}{3}h^2)$$

Hence

$$(a,x) = \wp(\sqrt{2}\imath t) + \frac{h}{3},$$

where $\wp$ is the Weirstrass function. We finally deduced that the behavior of the particle on the sphere under the action of the potential field of force with potential $U = (a,x)$ can be describe by the differential equations

$$\ddot{x}_j + (3\wp(\sqrt{2}\imath t) - h)x_j = -a_j, \quad j = 1, 2, .., N$$

Example 3.3
We shall apply the A-model in the well known Appel mechanical system [Appell]. Let L_0, L_1 are the functions:

$$\begin{cases} L_0 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - gz \\ L_1 = \dot{z} - ar = 0, \quad r = \sqrt{\dot{x}^2 + \dot{y}^2} \end{cases}$$

where a, g are nonnegative constants.

We obtain that for this case the matrix W and $E(L)$ take the form respectively:

$$W = \begin{pmatrix} -\frac{a\dot{x}}{r} & -\frac{a\dot{y}}{r} & 1 \\ \dot{x} & \dot{y} & \dot{z} \\ \partial_x f & \partial_y f & \partial_z f \end{pmatrix},$$

$$detW = (1+a^2)(\dot{y}\partial_x f - \dot{x}\partial_y f) \neq 0,$$

and

$$E(L) = \begin{pmatrix} \dot{y}q & -\dot{x}q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $q = \frac{-a(\ddot{x}\dot{y} - \ddot{x}\dot{y})}{r^3}$.

Hence the matrix A is the following

$$A^T = \frac{1}{detW}\begin{pmatrix} \dot{y}q(\dot{y}\partial_z f - \dot{z}\partial_y f) & -\dot{x}q(\dot{y}q(\dot{y}\partial_z f - \dot{z}\partial_y f) & 0 \\ \dot{y}q(\dot{z}\partial_x f - \dot{x}\partial_z f) & -\dot{x}q(\dot{z}\partial_x f - \dot{x}\partial_z f) & 0 \\ \dot{y}q(\dot{x}\partial_y f - \dot{y}\partial_x f) & -\dot{x}q(\dot{x}\partial_y f - \dot{y}\partial_x f) & 0 \end{pmatrix}$$

The equations of motion and transpositional relations as a consequence are the following

$$\begin{cases} \ddot{x} = -\frac{a\dot{x}}{r}\dot{\lambda} \\ \ddot{y} = -\frac{a\dot{y}}{r}\dot{\lambda} \\ \ddot{z} = -g + \dot{\lambda} \end{cases}$$

and

$$\begin{cases} \delta\dfrac{dz}{dt} - \dfrac{d\delta\, z}{dt} = -\dfrac{q}{detW}(\dot{y}\partial_z f - \dot{z}\partial_y f)(\dot{y}\delta x - \dot{x}\delta y) \\ \delta\dfrac{dy}{dt} - \dfrac{d\delta\, y}{dt} = \dfrac{q}{detW}(\dot{z}\partial_x f - \dot{x}\partial_z f)(\dot{y}\delta x - \dot{x}\delta y) \\ \delta\dfrac{dx}{dt} - \dfrac{d\delta\, x}{dt} = q(\dot{y}\delta x - \dot{x}\delta y) \end{cases}$$

From this example we observe that the independent variations produce the nonzero transpositional relations.

By derivation the constraints $\dot{z} - a\, r = 0$ we deduced that

$$\dot{\lambda} = \frac{g}{1+a^2},$$

hence the equations of motion are

$$\begin{cases} \ddot{x} = \dfrac{-ag}{1+a^2}\dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ \ddot{y} = \dfrac{-ag}{1+a^2}\dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ \ddot{z} = -\dfrac{a^2 g}{1+a^2} \end{cases} \tag{3.15}$$

A great many investigations have been devoted to the derivation of the equations of motion of mechanical systems with nonlinear nonholonomic constraints (see for instance [Chetaev, Hamel1, Novoselov]). The works of these authors do not contain examples of systems with nonlinear ideal nonholonomic constraints differing essentially from the example given by Appell in 1911.

Example 3.4

We shall deduce the equation of motion and transpositional relations for a constrained particle in R^3 which is under the action of the potential field of force with potential $U = U(z)$.

Let be the constraints L_1 :

$$L_1 = \dot{x} - a(z)\dot{y} = 0.$$

By choosing the function $f = z$, we easily deduced that the matrix W and $E(L)$ take the form respectively:

$$W = \begin{pmatrix} 1 & -a(z) & 0 \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & 1 \end{pmatrix}, \quad detW = (1 + a^2(z))\dot{y}$$

and

$$E(L) = \begin{pmatrix} 0 & -\partial_z a(z)\dot{z} & \partial_z a(z)\dot{y} \\ 0 & 0 & \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the matrix A :

$$A^T = \frac{1}{(1+a^2(z))\dot{y}} \begin{pmatrix} 0 & 0 & 0 \\ -\partial_z a(z)\dot{y} & \partial_z a(z)\dot{x} & 0 \\ \partial_z a(z)\dot{y}^2 & -\partial_z a(z)\dot{x}\dot{y} & 0 \end{pmatrix}$$

The equations of motion and transpositional relations in this case are:

$$\begin{cases} \ddot{x} = \dot{\lambda} \\ \ddot{y} = -a(z)\dot{\lambda} \\ \ddot{z} = \partial_z U(z) \end{cases}$$

and

$$\begin{cases} \delta \dfrac{dx}{dt} - \dfrac{d\delta\, x}{dt} = -\dfrac{\partial_z a(z)}{1+a^2(z)}(\dot{y}\delta z - \dot{z}\delta y) \\ \delta \dfrac{dy}{dt} - \dfrac{d\delta\, y}{dt} = -\dfrac{a(z)\partial_z a(z)}{1+a^2(z)}(\dot{y}\delta z - \dot{z}\delta y) \\ \delta \dfrac{dz}{dt} - \dfrac{d\delta\, z}{dt} = 0 \end{cases}$$

From the above results the following question arise:

Can be chosen the generalized coordinates in such a way that for nonholonomic systems the independent variations generates nonzero transpositional relations ?.

The positive answer to this question we obtain in particular for the so called *Chapliguin system* and for the generalization of such system which we proposed below.

In view of importance of the nonholonomic Chapliguin system we devote the following apartat to apply the A-model to study such class of system.

4. NONHOLONOMIC CHAPLIGUIN SYSTEMS .

It was pointed out by Chapliguin [Chap] that in many conservative nonholonomic systems the generalized coordinates

$$x^1, x^2, ..x^{S_1}, y^1, y^2, ..., y^{S_2}$$

can be choosen in such a way that the variations of the S_2 coordinates

$$y^1, y^2, ..., y^{S_2}$$

may be assumed to be independent and the remaining $S_2 = N - S_1$ coordinates occur neither in the coefficients a^j_k of the equations of the non-integrable kinematic constraints, written in the form

$$\dot{x}^j = \sum_{k=1}^{S_2} a^j_k(y^1, y^2, ..., y^{S_2})\dot{y}^k, \quad j = 1, 2, .., S_1 \tag{4.1}$$

nor in the expressions for the Lagrange function, written down without allowance for the constraints (4.1).

Classically, a mechanical system with Lagrangian

$$L_0(y^1, y^2, ..., y^{S_2}, \dot{x}^1, \dot{x}^2, ..., \dot{x}^{S_1}, \dot{y}^1, \dot{y}^2, ..., \dot{y}^{S_2}), \quad S_1 + S_2 = N$$

subject to S_1 linear nonholonomic constraints , is said to be a Chaplyguin type if the constraints can be written in the form (4.1).

4.1 GENERALIZED VORONES-CHAPLIGUIN MECHANICAL SYSTEM.

We apply the A-mathematical model for the more general case when the Lagrangian function L_0 :

$$L_0 = L_0(x, y, \dot{x}, \dot{y}),$$

and the ideal constraints are the following:

$$L_\alpha = \dot{x}^\alpha - \Phi^\alpha(u^{(1)}, u^{(2)}, ..., u^{(M)}, x, y, \dot{y}) = 0, \quad \alpha = 1, 2, .., S_1, \tag{4.2}$$

where

$$x = (x^1, x^2, ..., x^{S_1}), \quad y = (y^1, y^2, ..., y^{S_2}),$$

and

$$u^{(j)} = (u_1^{(j)}, u_2^{(j)}, ..., u_{S_3}^{(j)}), \quad j = 1, 2, .., M,$$

are certain real parameters.

First we shall study the subcase when the constraints do not depend on the parameters $u^{(j)}$, $j = 1, .., M$, i.e., we shall suppose that

$$L_\alpha = \dot{x}^\alpha - \Phi^\alpha(x, y, \dot{y}) = 0, \quad \alpha = 1, 2, .., S_1, \tag{4.3}$$

It is evident from the form of the constraints equations that the virtual variations $\delta y^j, \quad j = 1, 2, .., S_2$ may be assumed to be independent. The remaining variations

$$\delta x^j, \quad j = 1, 2, .., S_1$$

can be expressed in terms of them by the relations (Chetaev's conditions)

$$\theta_\alpha(\delta) \Longleftrightarrow \delta x^\alpha - \sum_{j=1}^{S_2} \frac{\partial L_\alpha}{\partial \dot{y}^j} \delta y^j = 0, \quad \alpha = 1, 2, .., S_1, \tag{4.4}$$

To construct the matrix W in this case we first determine the functions

$$L_{S_1+j} = \dot{y}^j, \quad j = 1, 2, .., S_2, ,$$

Clearly, in this case the matrix W can be represented as follow

$$
W = \begin{pmatrix}
1 & 0 & \dots & 0 & 0 & 0 & a_1^1 & \dots & a_{S_2}^1 \\
0 & 1 & \dots & 0 & 0 & 0 & a_1^2 & \dots & a_{S_2}^2 \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \vdots & \dots & \vdots & 1 & a_1^{S_1} & \dots & a_{S_2}^{S_1} \\
0 & 0 & \vdots & \dots & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots\vdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{4.5}
$$

$$
a_j^\alpha = \frac{\partial L_\alpha}{\partial \dot{y}^j}
$$

Denoting by $E(L)$ the matrix:

$$
E(L) = \begin{pmatrix}
E_1(L_1) & E_2(L_1) & \dots & E_N(L_1) \\
E_1(L_2) & E_2(L_2) & \dots & E_N(L_2) \\
\vdots & \vdots & \dots & \vdots \\
E_1(L_{S_1}) & E_2(L_{S_1}) & \dots & E_N(L_{S_1}) \\
0 & 0 & \dots & 0 \\
\vdots & \vdots & \dots & \vdots \\
0 & 0 & \dots & 0
\end{pmatrix} \tag{4.6}
$$

From (2.16)+ (4.5)+(4.6) we deduce that the matrix A in this case is the following

$$
A = [E(L)]^T \tag{4.7}
$$

We now turn to the derivation from (2.17)+(4.5) the dynamical equations and the transpositional relations are such that

$$
\begin{cases}
E_j(L_0) = \displaystyle\sum_{\alpha=1}^{S_1} E_j(L_\alpha) \frac{\partial L_0}{\partial \dot{x}^\alpha} + \dot{\lambda}_j, \quad j = 1, 2, .., S_1, \\
E_k(L_0) = \displaystyle\sum_{\alpha=1}^{S_1} (E_k(L_\alpha) \frac{\partial L_0}{\partial \dot{x}^\alpha} + \dot{\lambda}_\alpha \frac{\partial L_\alpha}{\partial \dot{y}^k}), \quad k = 1, 2, .., S_2
\end{cases} \tag{4.8}
$$

$$
\begin{cases}
\delta \dfrac{dx^\alpha}{dt} - \dfrac{d\delta\, x^\alpha}{dt} = \displaystyle\sum_{k=}^{S_2} (\sum_{j=1}^{S_1} E_j(L_\alpha) \frac{\partial L_j}{\partial \dot{y}^k} + E_k(L_\alpha)) \delta y^k, \quad \alpha = 1, 2, .., S_1 \\
\delta \dfrac{dy^m}{dt} - \dfrac{d\delta\, y^m}{dt} = 0, \quad m = 1, 2, ..., S_2
\end{cases} \tag{4.9}
$$

As we can observe the independent virtual variations $\delta y^j, \quad j = 1, 2, ..., S_2$ for the system with the ideal constraints (4.3) produce the zero transpositional relations.

Now we introduce the following definition

Definition 4.1

The mechanical system with the equations of motion (4.8) and transpositional relations (4.9) we call the *Generalized Vorones-Chapliguin Mechanical System.*

Below we establish the relation between the equation of motion of an electromechanical system with constraints and equation of motion for generalized Vorones- Chapliguin mechanical system.

Is well known that the equation of an electromechanical system can be written in the form of Lagrange equations of the second kind with Lagrangian function $\mathcal{L}$ which can be represented as follow

$$\mathcal{L} = L_{el} + L_{mech}$$

where L_{el} is a generating function for the mechanical forces of electromagnetic origin and L_{mech} is the Lagrangian function of a discrete mechanical system [Gaponov].

By following theses ideas we shall represented the Lagrangian function L_0 in the form:

$$L_0 = \Gamma(x, y, \dot{y}) + L^*(x, y, \dot{x}, \dot{y}), \tag{4.10}$$

where Γ and L^* are functions such that

$$\begin{cases} E_j(\Gamma) = \nu \sum_{\alpha=1}^{S_1} E_j(L_\alpha) \dfrac{\partial L_0}{\partial \dot{x}^\alpha}, \quad j = 1, 2, .., S_1, \\ E_k(\Gamma) = \nu \sum_{\alpha=1}^{S_1} E_k(L_\alpha) \dfrac{\partial L_0}{\partial \dot{x}^\alpha} \quad k = 1, 2, .., S_2 \end{cases} \tag{4.11}$$

$$\begin{cases} E_j(L^*) = (1 - \nu) \sum_{\alpha=1}^{S_1} E_j(L_\alpha) \dfrac{\partial L_0}{\partial \dot{x}^\alpha} + \dot{\lambda}_j, \quad j = 1, 2, .., S_1, \\ E_k(L^*) = (1 - \nu) \sum_{\alpha=1}^{S_1} E_k(L_\alpha) \dfrac{\partial L_0}{\partial \dot{x}^\alpha} + \sum_{\alpha=1}^{S_1} \dot{\lambda}_\alpha \dfrac{\partial L_\alpha}{\partial \dot{y}^k}, \quad k = 1, 2, .., S_2 \end{cases} \tag{4.12}$$

where ν is certain function.

Example 4.1

By using this model we shall deduced the differential equations which describe the behavior of the Apell system given in the example 3.1.

For this case we have

$$\begin{cases} L^* = \dfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - gz, \\ L_1 = \dot{z} - ar = 0, \quad r = \sqrt{\dot{x}^2 + \dot{y}^2} \\ L_2 = \dot{x} \\ L_3 = \dot{y} \end{cases}$$

In virtue of theses relations and by considering example 3.3, we have that the equations (4.12) in this case are the following :

$$\begin{cases} \ddot{z} = -g + \dot{\lambda}_1 \\ \ddot{y} + \dfrac{a^2\dot{x}(1-\nu)}{r^2}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) = -\dfrac{a\dot{\lambda}_1}{r}\dot{y} \\ \ddot{x} + \dfrac{a^2\dot{y}(1-\nu)}{r^2}(\dot{y}\ddot{x} - \dot{x}\ddot{y}) = -\dfrac{a\dot{\lambda}_1}{r}\dot{x} \\ \dot{z} = ar \end{cases}$$

By solving with respect to $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ we obtain exactly the equations (3.17). It is interesting to observe that in this case the matrix (2.20) takes the form:

$$G = \begin{pmatrix} 1 + \frac{a^2(1-\nu)\dot{y}^2}{r^2} & -\frac{a^2(1-\nu)\dot{x}\dot{y}}{r^2} \\ -\frac{a^2(1-\nu)\dot{x}\dot{y}}{r^2} & 1 + \frac{a^2(1-\nu)\dot{y}^2}{r^2} \end{pmatrix},$$

where $detG = 1 + a^2(1-\nu)$

The transpositional relations in this case are

$$\begin{cases} \delta\dfrac{dz}{dt} - \dfrac{d\delta\, z}{dt} = \dfrac{-a(\ddot{x}\dot{y} - \ddot{x}\dot{y})}{r^3}(\dot{y}\delta x - \dot{x}\delta y) \\ \delta\dfrac{dy}{dt} - \dfrac{d\delta\, y}{dt} = 0 \\ \delta\dfrac{dx}{dt} - \dfrac{d\delta\, x}{dt} = 0 \end{cases}$$

Example 4.2

Chapliguin-Carathodory's sleigh

We shall now analyze one of the most classical nonholonomic systems : Chapliguin-Carathodory's sleigh [NF]. The idealized sleigh is a body that has three points of contact with the plane. Two of them slide freely but the third, A, behaves like a knife edge subjected to a constraining force $\mathbf{R}$ which does not allow transversal velocity. More precisely, let yoz be an inertial frame and $\xi\, A\eta$ a frame moving with the sleigh. Take as generalized coordinates the Cartesian coordinates of the center of mass C of the sleigh and the angle x between the y and the ξ axis. The reaction force $\mathbf{R}$ against the runners is exerted laterally at the point of application A in such a way that the η component of the velocity is zero. Hence, one has the constrained system $\mathcal{M}$ with the configuration space $X = S^1 \times \mathrm{R}^2$, with the kinetic energy

$$T = \frac{m}{2}(\dot{y}^2 + \dot{z}^2) + \frac{I_c}{2}\dot{x}^2,$$

and with the constraint

$$a\dot{x} + \sin x\dot{y} - \cos x\dot{z} = 0,$$

where m is the mass of the system and I_c is the moment of inertia about a vertical axis through C and $a = |AC|$. Observe that the "javelin" (or arrow or Chapliguin's skate) is a particular case of this mechanical system and can be obtained when $a = 0$. We study only the case when the motion is by inertia and $a = 1$.

To apply the A-model for this system, first we introduce the 1-form Θ_j, $j = 1, 2, 3$ in such a way that

$$\Theta_1 = dx + \sin x\, dy - \cos x\, dz$$
$$\Theta_2 = dy$$
$$\Theta_3 = dz$$

The matrix W, $E(L)$ in this case are the following

$$W = \begin{pmatrix} 1 & \sin x & \cos x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad detW = 1,$$
$$E(L) = \begin{pmatrix} -\cos x\, \dot{y} - \sin x\, \dot{z} & \cos x \dot{x} & \sin x\, \dot{x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By introducing the function L^* :

$$L^* = \frac{m}{2}(\dot{y}^2 + \dot{z}^2) + \frac{I_C}{2}\dot{x}^2,$$

we deduced that for the sleigh the equation (4.12) and transpositional relations take the form respectively :

$$\begin{cases} J_C \ddot{x} = \dot{\lambda} - J_C(1-\nu)\dot{x}(\dot{y}\cos x + \dot{z}\sin x) \\ m\ddot{y} = \sin x\, \dot{\lambda} + J_C(1-\nu)\dot{x}^2 \cos x \\ m\ddot{z} = -\cos x\, \dot{\lambda} + J_C(1-\nu)\dot{x}^2 \sin\, x \end{cases}$$

and

$$\begin{cases} \delta\dfrac{dx}{dt} - \dfrac{d\delta\, x}{dt} = -(\dot{y}\cos x + \dot{z}\sin x)\delta x + \dot{x}\cos x\delta y + \dot{x}\sin\, x\delta z = \dot{z}\delta y - \dot{y}\delta z \\ \delta\dfrac{dz}{dt} - \dfrac{d\delta\, z}{dt} = 0 \\ \delta\dfrac{dy}{dt} - \dfrac{d\delta\, y}{dt} = 0 \end{cases}$$

where we consider that the constraint is ideal, i.,e.,

$$\delta x = \cos x\delta z - \sin x\delta y$$

4.1 GENERALIZED CHAPLIGUIN MECHANICAL SYSTEM.

The important subcase we obtain when the function ν, L_0 and the constraints are such that:

$$
(4.13) \qquad \begin{cases} \nu = 1 \\ L_0 = \Gamma(y, \dot{y}) + L^*(y, \dot{x}, \dot{y}) \\ L_\alpha = \dot{x}^\alpha - \Phi^\alpha(y, \dot{y}), \quad \alpha = 1, .., S_1 \end{cases}
$$

then equations (4.11), (4.12) can be rewritten as follows:

$$
(4.14) \qquad E_k(\Gamma) = \sum_{\alpha=1}^{S_1} E_k(L_\alpha) \frac{\partial L_0}{\partial \dot{x}^\alpha} \quad k = 1, 2, .., S_2
$$

$$
(4.15) \qquad \begin{cases} E_j(L^*) = \dot{\lambda}_j, \quad j = 1, 2, .., S_1, \\ E_k(L^*) = \sum_{\alpha=1}^{S_1} \dot{\lambda}_\alpha \frac{\partial L_\alpha}{\partial \dot{y}^k}, \quad k = 1, 2, .., S_2 \end{cases}
$$

By considering that, under given conditions (4.13)

$$
E_j(L^*) = \frac{d}{dt} \frac{\partial L_0}{\partial \dot{x}^j}, \quad j = 1, .., S_2
$$

we easily deduced that the above equations take the form

$$
E_k(L^*) = \sum_{\alpha=1}^{S_1} \frac{d}{dt} \frac{\partial L_0}{\partial \dot{x}^\alpha} \frac{\partial L_\alpha}{\partial \dot{y}^k}
$$

Now we shall affix an tilde to an expression in which the generalized velocities

$$
\dot{x}^1, \dot{x}^2, .., \dot{x}^{S_1}
$$

have been eliminated by means of the constraints equations (4.13), i.e.,

$$
\tilde{L} = L^*((y, \dot{x}, \dot{y})|_{\dot{x}^\alpha = \Phi^\alpha(y, \dot{y})}.
$$

The following relations follow immediately

$$
\begin{cases} \dfrac{\partial \tilde{L}}{\partial \dot{y}^j} = \dfrac{\partial L^*}{\partial \dot{y}^j} - \sum_{\alpha=1}^{S_1} \dfrac{\partial L^*}{\partial \dot{x}^\alpha} \dfrac{\partial L_\alpha}{\partial \dot{y}^j} \\ \dfrac{\partial \tilde{L}}{\partial y^j} = \dfrac{\partial L^*}{\partial y^j} - \sum_{\alpha=1}^{S_1} \dfrac{\partial L^*}{\partial \dot{x}^\alpha} \dfrac{\partial L_\alpha}{\partial y^j} \end{cases}
$$

By virtue of these relations, we have

$$E_k(L^* - \tilde{L}) = \sum_{\alpha=1}^{S_1}(\frac{d}{dt}\frac{\partial L_0}{\partial \dot{x}^\alpha}\frac{\partial\, L_\alpha}{\partial \dot{y}^k} + \frac{\partial L_0}{\partial \dot{x}^\alpha}E_k(L_\alpha)). \tag{4.16}$$

Proposition 4.1

The equations (4.14), (4.15) coincide if

$$\Gamma = -\tilde{L} = -L^*|_{\dot{x}^\alpha = \Phi^\alpha(y,\dot{y})} \tag{4.17}$$

In fact, if in (4.8)+(4.13) we determine

$$\dot{\lambda}_j = \frac{d}{dt}\frac{\partial L_0}{\partial \dot{x}^j}$$

we obtain the equations (4.16) if we Γ determine by the formula (4.17).

The transpositional relations (4.9) under the restrictions (4.13) take the form

$$\begin{cases} \delta\dfrac{dx^\alpha}{dt} - \dfrac{d\delta\, x^\alpha}{dt} = \sum_{k=}^{S_2}(E_k(L_\alpha))\delta y^k, \quad \alpha = 1,2,..,S_1 \\ \delta\dfrac{dy^m}{dt} - \dfrac{d\delta\, y^m}{dt} = 0, \quad m = 1,2,...,S_2 \end{cases} \tag{4.18}$$

Definition 4.2

The mechanical system with equation of motion (4.14), or, (4.15) and transpositional relations (4.18) we call the *Generalized Chapliguin System*

Let us denote by $\mathcal{R}$ the following function

$$\begin{cases} \mathcal{R} = L_0 - \sum_{\alpha=1}^{S_1}\dfrac{\partial L_0}{\partial \dot{x}^\alpha}\, L_\alpha \equiv R - R_0 \\ R = L^* - \sum_{\alpha=1}^{S_1}\dfrac{\partial L^*}{\partial \dot{x}^\alpha}\, \dot{x}^\alpha \\ R_0 = R|_{\dot{x}^\alpha = \Phi^\alpha(y,\dot{y})},\ \alpha = 1,2,...,S_1 \end{cases} \tag{4.19}$$

Corollary 4.1

From the equations (4.16) we obtain the system

$$E_k(\mathcal{R}) = \frac{d}{dt}(L_\alpha)\,\partial_{\dot{y}^k}(\partial_{\dot{x}^\alpha}L_0) - \partial_{y^k}(\partial_{\dot{x}^\alpha}L_0)\,L_\alpha, \quad k = 1,2,..,S_2, \quad \alpha = 1,2,..,S_1$$

The proof of this corollary follow from the fact that

$$
\begin{aligned}
E_k(L_\alpha\,\partial_{\dot{x}^\alpha}L_0) =&\partial_{\dot{x}^\alpha}L_0\,E_k(L_\alpha) + \partial_{\dot{x}^\alpha}L_\alpha\frac{d}{dt}(\partial_{\dot{x}^\alpha}L_0)\\
&+\frac{d}{dt}(L_\alpha)\,\partial_{\dot{y}^k}(\partial_{\dot{x}^\alpha}L_0) - \partial_{y^k}(\partial_{\dot{x}^\alpha}L_0)\,L_\alpha,
\end{aligned}
$$

Proposition 4.1
The equations (4.16) are equivalent to the system

$$
\begin{cases} E_k(\mathcal{R}) = 0, \quad k = 1, 2, .., S_2 \\ L_\alpha = \dot{x}^\alpha - \Phi^\alpha(y, \dot{y}) = 0, \quad \alpha = 1, 2, .., S_1 \end{cases} \tag{4.20}
$$

The proof is easy to obtain from the above corollary.
Corollary 4.2
Let L^* be a function:

$$
L^* = \frac{1}{2}\sum_{k,j=1}^{S_2} G_{kj}(y)\dot{y}^k\dot{y}^j + \frac{1}{2}\sum_{\alpha,\beta=1}^{S_1} G_{\alpha\beta}(y)\dot{x}^\alpha\dot{x}^\beta + \sum_{\alpha=1}^{S_1}\sum_{k=1}^{S_2} G_{kj}(y)\dot{x}^\alpha\dot{y}^k + U(y)
$$

Then the function $\mathcal{R}$:

$$
\mathcal{R} = R - R_0 = -\frac{1}{2}\sum_{\alpha,\beta=1}^{S_1} G_{\alpha\beta}(y)(\dot{x}^\alpha - \Phi^\alpha)(\dot{x}^\beta - \Phi^\beta)
$$

Example 4.3
For the constrained particle in R^3 study in the example 3.3 we have that

$$
\begin{cases} L_1 =\dot{x} - a(z)\dot{y} \\ L^* =\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ L^*|_{\dot{x}-a(z)\dot{y}=0} =\frac{1}{2}((1 + a^2(z))\dot{y}^2 + \dot{z}^2) \end{cases}
$$

Hence,

$$
\mathcal{R} = -\frac{1}{2}(\dot{x} - a(z)\dot{y})^2.
$$

The very important subcase of the Chapliguin system is the case when the constraints are linear, i.e., when the functions L_α

$$
L_\alpha = \dot{x}^\alpha - \sum_{j=1}^{S_2} a_j^\alpha(y)\dot{y}^j, \quad \alpha = 1, 2, .., S_1, \quad S_1 + S_2 = N,
$$

From (4.14) we deduced the following differential equations:

$$E_k(\Gamma) = \sum_{j=1}^{S_1}\sum_{r=1}^{S_2}(\partial_{y^k}\alpha_r^j - \partial_{y^r}\alpha_k^j)\dot{y}^r\frac{\partial L^*}{\partial \dot{x}^j}, \quad k = 1, 2, .., S_2$$

which are the equations which Chapliguin published in the Proceeding of the Society of the Friends of Natural Science in 1897 [Chapliguin].

Clearly, the equations (4.14) are an extension of the Chapliguin model for the nonlineal constraints.

Example 4.4

In [Appell] a study is made of the following nonholonomic system: A weight of mass m hangs on a thread which passes around the pulleys and is wound round the drum of radius a. The drum is fixed to a wheel of radius b which rolls without sliding on a horizontal plane, touching it at the point B with the coordinates $(x_B,\, y_B, 0)$. The legs of the frame that support the pulleys and keep the plane of the wheel verical slide on the horizontal plane without friction. Let θ be the angle between the plane of the wheel and the Ox axis; φ the angle of the rotation of the wheel in its own plane; and (x, y, z) the coordinates of the mass m. Clearly,

$$\dot{z} = a\dot{\varphi}$$

The coordinates of the point B and the coordinates of the centre mass are related as follows

$$x = x_B + \rho\cos\theta, \quad y = y_B + \rho\sin\theta$$

The condition of rolling without sliding leads to the equations of nonholonomic constraints:

$$\dot{x_B} = b\cos\theta\dot{\varphi}, \quad \dot{y_B} = b\sin\theta\dot{\varphi}$$

By neglecting the mass of the frame and the mass of the wheel and supposing that $m = 1$, we obtain the following expression for the function L^* of the system:

$$L^* = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - gz$$

By introducing the following notations

$$\begin{aligned} L_1 &= \dot{x} - b\cos\theta\dot{\varphi} - \varrho\sin\theta\dot{\theta} \\ L_2 &= \dot{y} - b\sin\theta\dot{\varphi} + \varrho\cos\theta\dot{\theta} \\ L_3 &= \dot{z} - a\dot{\varphi} \end{aligned}$$

where

$$\begin{aligned} &x_1 = x,\, x_2 = y,\, x_3 = z \\ &y_1 = \theta,\, y_2 = \varphi \end{aligned}$$

The differential equations (4.15), or, what is the same, (4.14)+(4.17), describing the behavior of the system with the Lagrangian function L^* and constraints

$$L_j = 0, \quad j = 1, 2, 3$$

can be rewritten as follows respectively:

$$\begin{cases} \ddot{x} = \dot{\lambda}_1 \\ \ddot{y} = \dot{\lambda}_2 \\ \ddot{z} = -g - a\dot{\lambda}_3 \\ \rho(\dot{\lambda}_1 \sin\theta - \dot{\lambda}_2 \cos\theta) = 0 \\ b\dot{\lambda}_1 \cos\theta + b\dot{\lambda}_2 \sin\theta + a\dot{\lambda}_3 = 0 \end{cases}$$

and

$$\begin{cases} \rho^2\ddot{\theta} + b\rho\dot{\theta}\dot{\varphi} = 0 \\ (a^2+b^2)\ddot{\varphi} - b\rho\dot{\theta}^2 = -ga \end{cases}$$

which, together with the equations of the linear nonholonomic constraints describe the inertial motion of the nondegenerate system.

From the above, after straightforward calculations we deduced the differential equations:

$$\begin{cases} \ddot{x} = -\frac{a}{b}\cos\theta\dot{\lambda}_3 = -\frac{a\dot{x}}{(a^2+b^2)\sqrt{\dot{x}^2+\dot{y}^2}} + \rho A_1 + \rho^2 A_2 + ... \\ \ddot{y} = -\frac{a}{b}\sin\theta\dot{\lambda}_3 = -\frac{a\dot{y}}{(a^2+b^2)\sqrt{\dot{x}^2+\dot{y}^2}} + \rho B_1 + \rho^2 B_2 + .. \\ \ddot{z} = -g - a\dot{\lambda}_3 = -\frac{a^2 g}{a^2+b^2} + \rho D_1. \end{cases}$$

where $A_1, A_2, ..., B_1, .., B_2, .., D_1$ are certain functions.

Hamel in [Hamel2] obtained the equations of motion of a nonholonomic system with nonlinear constraints from the above equations by making the passage to the limit $\rho \to 0$. For $\rho \to 0$ we obtain from the above the equations (3.15) and

$$\begin{cases} \dot{\theta} = 0 \\ (a^2+b^2)\ddot{\varphi} = -ga \end{cases}$$

The system with nonlinear nonholonomic constraints considered by Appell and Hamel is obtained from a nonholonomic system with linear constraints by means of the passage to the limit $\rho \to 0$. However, as a result of this limiting process, the order of the system of differential is reduced, i.e., they become degenerate.

For $\rho \neq 0$ we obtain, after integration :

$$\begin{cases} \theta = C_1 \int \exp(-\frac{2b}{\rho}\varphi(t))\, dt + C_0 \\ \int \frac{d\varphi}{\sqrt{\frac{\rho b C_1}{a^2+b^2}\exp(-\frac{2b}{\rho}\varphi) - ga\varphi + C_2}} = t - t_0 \end{cases}$$

The transpositional relations for the nondegenerate system are the following:

$$\begin{cases} \delta\dfrac{dx}{dt} - \dfrac{d\delta\, x}{dt} = b\sin\theta(\dot\theta\delta\varphi - \dot\varphi\delta\theta) \\ \delta\dfrac{dy}{dt} - \dfrac{d\delta\, y}{dt} = -b\cos\theta(\dot\theta\delta\varphi - \dot\varphi\delta\theta) \\ \delta\dfrac{dz}{dt} - \dfrac{d\delta\, z}{dt} = 0 \end{cases}$$

Clearly, these relations are completely different from the relations deduced for degenerate system (see example (3.3)).

The dynamic of the nondegenerate system was study in [NF1]. The authors proved that the motion of the degenerate system differ essentially from the limiting motion and, hence, they deduced that the example of a nonholonomic system with nonlinear constraints is incorrect

4.3 ROUTH-CHAPLIGUIN MECHANICAL SYSTEM.

Now we shall study the case when the constraints (4.2) are such that

(4.21) $$L_\alpha = \dot x^\alpha - \Phi^\alpha(u^{(1)}, y, \dot y) = 0, \quad \alpha = 1, 2, .., S_1$$

where $u^{(1)} = (u_1, u_2, ..., u_{S_1})$ is a constant vector such that

(4.22) $$\frac{\partial L^*}{\partial \dot x^j} = u^{(1)}_j = const., \quad j = 1, 2, .., S_1$$

Under these restrictions the Lagrangian multiplier λ_j, $j = 1, 2, .., S_1$ are constants.

The relations (4.21) are obtained by solving (4.22) with respect to $\dot x^1, \dot x^2, ...\dot x^{S_1}$.

Definition 4.3

The mechanical system for which (4.22) holds we call the *Routh -Chapliguin System,* and the functions

$$R = L^*(y, \dot x, \dot y) - \sum_{\alpha=1}^{S_1} u^{(1)}_\alpha \dot x^\alpha$$

and

$$R_0 = R|_{\dot x^\alpha = \Phi^\alpha(u^{(1)}, y, \dot y)}$$

are the *Routh functions.*

From (4.14)+(4.22) and (4.15)+(4.22) we deduced the Routh differential equations

$$E_k(R) = 0 \quad k = 1, 2, ..., S_2$$

or, what is the same

$$E_k(R_0) = 0, \quad k = 1, 2, .., S_2$$

Example 4.5
We shall study the particle:

$$L^* = \frac{ml^2}{2}(\dot{y}^2 + \sin^2 y\dot{x}^2) - mgl\cos y$$

In this case we have

$$\begin{aligned}
&\frac{\partial L^*}{\partial \dot{x}} = u_1 = Const.\\
&R = \frac{k^2}{2}(\dot{y}^2 + \sin^2 y\dot{x}^2) - -u_1\dot{x} - mgl\cos y\\
&R_0 = R|_{k^2\sin^2 y\dot{x}=u_1} - mgl\cos y\\
&\mathcal{R} = R - R_0 = \frac{1}{2k^2\sin^2 y}(k^2\dot{x}\sin^2 y - u_1)^2
\end{aligned}$$

4.3 GAPONOV-CHAPLIGUIN MECHANICAL SYSTEM.

Now we suppose that in (4.2) we have a countable set on conta $S_1 = +\infty$ and the constraints are such that to

$$L_\alpha = \dot{x}^\alpha - \Phi^\alpha((u^{(1)}, u^{(2)}, ..., u^{(S_2)}, \dot{y}) = 0, \quad \alpha = 1, 2, ... \tag{4.23}$$

The equations (4.14) and (4.15) in this case are such that

$$\begin{aligned}
E_k(\Gamma) &= \sum_{\alpha=1}^{+\infty} \frac{d}{dt}\frac{\partial L_\alpha}{\partial \dot{y}^k}\frac{\partial L_0}{\partial \dot{x}^\alpha},\\
E_k(L^*) &= \sum_{\alpha=1}^{+\infty} \frac{d}{dt}\frac{\partial L_0}{\partial \dot{x}^\alpha}\frac{\partial L_\alpha}{\partial \dot{y}^k}
\end{aligned}$$

Clearly, the functions (4.19) and equations (4.20) in this case take the form respectively:

$$\begin{cases}
\mathcal{R} = L_0 - \sum_{\alpha=1}^{+\infty} \frac{\partial L_0}{\partial \dot{x}^\alpha} L_\alpha \equiv R - R_0\\
R = L^* - \sum_{\alpha=1}^{+\infty} \frac{\partial L^*}{\partial \dot{x}^\alpha}\dot{x}^\alpha\\
R_0 = R|_{\dot{x}^\alpha = \Phi^\alpha(y, \dot{y})},\ \alpha = 1, 2, ...
\end{cases}$$

$$\begin{cases}
E_k(\mathcal{R}) = 0,\\
L_\alpha = 0,\ \alpha = 1, 2, ...
\end{cases} \tag{4.24}$$

The transpositional relations (4.18)+(4.23) in this case are the following

$$(4.25) \qquad \begin{cases} \delta\dfrac{dx^\alpha}{dt} - \dfrac{d\delta\, x^\alpha}{dt} = \displaystyle\sum_{k=}^{+\infty} \dfrac{d}{dt}\dfrac{\partial L_\alpha}{\partial \dot y^k}\delta y^k, \quad \alpha = 1,2,..,S_1 \\ \delta\dfrac{dy^m}{dt} - \dfrac{d\delta\, y^m}{dt} = 0, \quad m = 1,2,...,S_2 \end{cases}$$

Definition 4.4

The mechanical system with equation of motion (4.24), and transpositional relations (4.25) we call the *Generalized Gaponov-Chapliguin System*

We now turn to the derivation of Gaponov's equation by applying the A-model.

Let $(x^1, x^2, ...; y^1, y^2, ..., y^{S_2})$ be the generalized coordinates of the system and let

$$L_0 = L_0(y^1, y^2, ..., y^{S_2}, \dot x^1, \dot x^2, \dot x^3, ...\dot y^1, \dot y^2, ...\dot y^{S_2})$$

be its Lagrange function, which does not explicitly contain the coordinates x. We shall assume that the system is subjected to a countable set of nonholonomic constraints of the form

$$(4.26) \qquad L_\alpha = \dot x^\alpha - \sum_{j=1}^{S_2} A_j^\alpha(u_j^{(1)}, u_j^{(2)}, ..., u_j^{(S_2)})\dot y^j = 0, \quad \alpha = 1,2,3,....,$$

By inserting (4.26) into the first equations of (4.24), after some calculations we obtain the differential equations which contain only $\Gamma(u^{(1)}, ...u^{(S_1)}, y, \dot y)$

$$E_j(\Gamma) + \frac{1}{\dot y^j}\sum_{m=1}^{S_2} \dot u_m^j \frac{\partial \Gamma}{\partial u_m^j} = 0, \quad j = 1,2,..,S_2$$

These equations with a slightly different notation were derived in 1953 by Gaponov [Gaponov].

The nonholonomic constraints which have the form of equations (4.26) are realized in a system containing three-dimensional conductors with sliding contacts.. The equations of motion of dynamical system with such constraints in the case of a finite number of degrees of freedom, as we observe above, were first derived by Chapliguin. Thus, electromechanical system with sliding contact belong to the class of nonholonomic system of Chapliguin. However, because of the fact that the Chapliguin's equations of motion contain the coefficients of the nonholonomic constraints, it is difficult to use these equations in the case of a system with a countable set of constraints. This is because Chapliguin's equation contain the original Lagrange function L^* and also the transformed function $\Gamma = L^*|_{\dot x^\alpha = \Phi^\alpha}, \quad \alpha = 1,2,...$ As a result, the equations of motion of electrical machines derived in this manner contain a countable set of coefficients of nonholonomic constraints.

In his investigation Gaponov [Gap] proposed equations of motion which have the advantage that, for applications to electromechanical system and, particularly, system containing three dimensional conductors with sliding contacts, they only contain a single function Γ, whose "electrical" part may be written down without knowledge of the function L^* and the equations of the nonholonomic constraints[NF].

Now we shall illustrate the A-model in the following example.

Example 4.6

Homogenous sphere on a horizontal plane rotating with a constant angular velocity Ω.

Let the Oz axis of the fixed system of coordinates $Oxyz$ coincide with the axis of rotation. Then, as the sphere moves on the plane, it has kinetic energy

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\kappa^2}{2}(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta),$$

where κ is the radius of gyration; the mass of the sphere is $m = 1$; x and y are the coordinates of the center of the sphere; and θ, ψ and θ are the Eulerian angles. Denoting by ω the instantaneous angular velocity of the sphere and by a its radius, we write down the condition of rolling without sliding of the sphere on the rotating plane:

$$\begin{aligned} \dot{x} - a(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi) + \Omega\, x = 0 \\ \dot{y} + a(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi) - \Omega\, y = 0 \end{aligned}$$

It is interesting to observe that theses constraints can be rewritten as follows

$$\begin{cases} \dot{u} - a\tilde{\omega}_y = 0 \\ \dot{v} + a\tilde{\omega}_x = 0 \end{cases}$$

where

$$\begin{cases} u = \cos(\Omega\, t)\, x - \sin(\Omega\, t)\, y \\ v = \cos(\Omega\, t)\, y + sin\,(\Omega\, t)\, y \\ \tilde{\omega}_y = \dot{\theta}\sin(\psi - \Omega\, t) - \dot{\varphi}\sin\theta\cos(\psi - \Omega\, t) \\ \tilde{\omega}_x = \dot{\theta}\cos(\psi - \Omega\, t) + \dot{\varphi}\sin\theta\sin(\psi - \Omega\, t) \\ \tilde{\omega}_z = \dot{\psi} - \Omega + \dot{\varphi}\cos\theta \end{cases}$$

Since we have three degrees of freedom and the coordinates x, y, θ and ψ all occur in the constraint equation and in the expression for kinetic energy, we have an example of a generalized Chapliguin system. To deduce the equations of motion we use the formula (4.11)+(4.12) with

$$L^* = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\kappa^2}{2}(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta)$$

By considering that in this case the matrix $E(L)$ is the following

$$E(L) = \begin{pmatrix} 0 & -\Omega & -a\cos\psi\omega_z & a\frac{d}{dt}(\sin\theta\cos\psi) & a\omega_x \\ \Omega & 0 & -a\sin\psi\omega_z & a\frac{d}{dt}(\sin\theta\sin\psi) & a\omega_y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where we introduce the well known notations

$$\begin{cases} \omega_x = \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi \\ \omega_y = \dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi \\ \omega_z = \dot{\psi} + \dot{\varphi}\cos\theta \end{cases}$$

After some calculations we obtain the following expression for the equations(4.12) and transpositional relations in this case

$$\begin{cases} \ddot{x} = \beta\Omega\dot{y} + \dot{\lambda}_1, \\ \ddot{y} = -\beta\Omega\dot{x} + \dot{\lambda}_2 \\ \kappa^2(\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta) = -\beta\, a((\dot{x}\cos\psi + \dot{y}\sin\psi)(\dot{\psi} + \dot{\varphi}\cos\theta)) + \cos\psi\dot{\lambda}_2 - \sin\psi\dot{\lambda}_1 \\ \kappa^2\frac{d}{dt}(\dot{\varphi} + \dot{\psi}\cos\theta) = a\beta\,(\dot{\theta}\cos\theta\cos\psi - \dot{\psi}\sin\theta\sin\psi)\dot{x} + a(\dot{\theta}\cos\theta\sin\psi + \dot{\psi}\sin\theta\cos\psi)\dot{y} + \\ \quad a(\sin\psi\sin\theta\dot{\lambda}_2 + \cos\psi\sin\theta\dot{\lambda}_1) \\ \kappa^2\frac{d}{dt}(\dot{\psi} + \dot{\varphi}\cos\theta) = a\beta\,((\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)\dot{x} + a(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\dot{y}) \end{cases}$$

where $\beta \equiv 1 - \nu$, and

$$\begin{cases} \delta\frac{dx}{dt} - \frac{d\delta\, x}{dt} = a(\cos\psi(\dot{\theta}\delta\tilde{\psi} - \dot{\tilde{\psi}}\delta\theta) + \sin\theta\sin\psi(\dot{\varphi}\delta\tilde{\psi} - \dot{\tilde{\psi}}\delta\varphi) + \cos\theta\cos\psi(\dot{\theta}\delta\varphi - \dot{\varphi}\delta\theta)) \\ \delta\frac{dy}{dt} - \frac{d\delta\, y}{dt} = a(\sin\psi(\dot{\theta}\delta\tilde{\psi} - \dot{\tilde{\psi}}\delta\theta) - \sin\theta\cos\psi(\dot{\varphi}\delta\tilde{\psi} - \dot{\tilde{\psi}}\delta\varphi) + \cos\theta\sin\psi(\dot{\theta}\delta\varphi - \dot{\varphi}\delta\theta)) \\ \delta\frac{d\theta}{dt} - \frac{d\delta\,\theta}{dt} = 0 \\ \delta\frac{d\varphi}{dt} - \frac{d\delta\,\varphi}{dt} = 0 \\ \delta\frac{d\psi}{dt} - \frac{d\delta\,\psi}{dt} = 0 \end{cases}$$

where

$$\tilde{\psi} = \psi - \Omega\, t$$
$$\delta\, x = a(\delta\theta\sin\psi - \delta\varphi\varphi\sin\theta\cos\psi)$$
$$\delta y = -a(\delta\theta\cos\psi + \delta\varphi\sin\theta\sin\psi)$$

For the Lagrangian multiplier it is easy to obtain the following expression:

$$\dot{\lambda}_1 = -\dot{y}\frac{\beta(\Omega\kappa^2 + a^2(\dot{\psi} + \dot{\varphi}\cos\theta) + \Omega\kappa^2}{a^2 + \kappa^2}$$
$$\dot{\lambda}_2 = \dot{x}\frac{\beta(\Omega\kappa^2 + a^2(\dot{\psi} + \dot{\varphi}\cos\theta) + \Omega\kappa^2}{a^2 + \kappa^2}$$

Using these values and notations, we write down the equations of motion for the sphere

$$\begin{cases} \dot{\omega}_x = \dfrac{a^2\beta(\Omega - \omega_z) + \Omega\, a}{a^2 + \kappa^2}\dot{x} \\ \dot{\omega}_y = \dfrac{a^2\beta(\Omega - \omega_z) + \Omega\, a}{a^2 + \kappa^2}\dot{y} \\ \kappa^2\dot{\omega}_z = -\beta\, a(\dot{x}\omega_x + \dot{y}\omega_y) \\ \ddot{x} = \dfrac{\beta(a^2\Omega - a^2\omega_z) - \Omega\kappa^2}{a^2 + \kappa^2}\dot{y} \\ \ddot{y} = -\dfrac{\beta(a^2\Omega - a^2\omega_z) - \Omega\kappa^2}{a^2 + \kappa^2}\dot{x} \end{cases}$$

By eliminating ω_x, ω_y by means of the equations of the nonholonomic constraints

$$\begin{cases} a\omega_x = \Omega\, x - \dot{y} \\ a\omega_y = \Omega\, y + \dot{x} \end{cases} \tag{4.27}$$

we obtain the equations

$$\begin{cases} \kappa^2\dot{\omega}_z = -\beta\Omega(x\dot{x} + y\dot{y}) \\ \ddot{x} = \dfrac{\beta(a^2\Omega - a^2\omega_z) - \Omega\kappa^2}{a^2 + \kappa^2}\dot{y} \\ \ddot{y} = -\dfrac{\beta(a^2\Omega - a^2\omega_z) - \Omega\kappa^2}{a^2 + \kappa^2}\dot{x} \end{cases} \tag{4.28}$$

which, together with the constraints equations, form a complete system for the determination ω_z, ω_x, ω_y and x, y as a function of the time.

From the first equation we deduce the first integral

$$\kappa^2\omega_z + \beta\frac{\Omega}{2}(x^2 + y^2) = C_1,$$

where C_1 is an arbitrary constant. By inserting ω_z into the equations respect to x and y we arrive at the equations:

$$\begin{cases} \ddot{x} = (\dfrac{\beta(a^2\Omega + a^2(\beta\frac{\Omega}{2}(x^2 + y^2) - C_1) - \Omega\kappa^2}{a^2 + \kappa^2})\dot{y} \equiv (-K_1 + K_2(x^2 + y^2))\dot{y} \\ \ddot{y} = -(\dfrac{\beta(a^2\Omega + a^2(\beta\frac{\Omega}{2}(x^2 + y^2) - C_1) - \Omega\kappa^2}{a^2 + \kappa^2})\dot{x} \equiv (K_1 - K_2(x^2 + y^2))\dot{x} \end{cases}$$

where

$$K_1 = \frac{\beta(a^2\Omega - a^2C_1) - \Omega\kappa^2}{a^2 + \kappa^2}$$

$$K_2 = \frac{a^2\beta\Omega}{2(a^2 + \kappa^2)}$$

The gyroscopic nature of the nonholonomy terms we can see directly from these equations. As a consequence we obtain a further two first integrals

$$\dot{x}^2 + \dot{y}^2 = C_2,$$
$$y\dot{x} - x\dot{y} + \frac{1}{2}\int (K_1 - K_2(x^2+y^2))d(x^2+y^2) = C_3$$

where C_1, C_2 are arbitrary constants.

By introducing the polar coordinates

$$x = r\cos\tau, \quad y = r\sin\tau,$$

the obtained first integral can reexpressed as follow

$$\begin{cases} \kappa^2\omega_z + \frac{1}{2}\beta\Omega\, r^2 = C_1 \\ \dot{r}^2 + r^2\dot{\tau}^2 = C_2 \\ r^2\dot{\tau} - (\frac{1}{2}K_1 r^2 - \frac{1}{4}K_2 r^4) = C_3 \end{cases}$$

which we apply to obtain for the determination the variables x, y, ω_x, ω_y, ω_z as a function of time:

$$\begin{cases} \dot{\tau} = (\frac{K_1 + 2C_3}{2r^2} - \frac{K_2 r^2}{4}) \\ r^2\dot{r}^2 = C_2 r^2 - r^4\dot{\varphi}^2 \equiv -R_4(r^2, C_1, C_2, C_3, \Omega) \end{cases}$$

where R_4 is a polynomial of degree four and such that

$$R_4(q, C_1, C_2, C_3, \Omega) = \frac{K_2^2}{16}q^4 - \frac{K_1K_2}{8}q^3 + (\frac{K_1^2}{4} + \frac{K_2}{2})q^2 - (K_1 + C_2)q - C_3^2$$

Clearly, the dependence of time require previously to solve the elliptic integral

$$\int \frac{dr^2}{\sqrt{R_4(r^2, C_1, C_2, C_3, \Omega)}} = \imath(t - t_0) \tag{4.29}$$

In [NF] the authors derive the following equations of motion for homogenous sphere on a horizontal plane rotating with a constant angular velocity Ω

$$\begin{cases} \dot{\omega}_x = \frac{a\Omega}{a^2+\kappa^2}(a\omega_y - \Omega\, y) \\ \dot{\omega}_y = -\frac{a\Omega}{a^2+\kappa^2}(a\omega_x - \Omega\, y) \\ \kappa^2\dot{\omega}_z = 0 \end{cases}$$

which together with the constraints equations, form a complete system for the determination of x, y, ω_x, ω_y, ω_z as functions of the time.

From the above equations it is easy to deduce the existence of the three first integrals

$$\omega_x - \frac{a\Omega}{a^2+\kappa^2}x = C_1$$
$$\omega_y - \frac{a\Omega}{a^2+\kappa^2}y = C_2$$
$$\omega_z = C_3$$

Eliminating ω_x, ω_y from the constraints, by means of the above relations, we arrive at the following equations for x and y :

$$\begin{cases} \ddot{x} + \dfrac{\kappa^2\Omega}{a^2+\kappa^2}\dot{y} = 0 \\ \ddot{y} - \dfrac{\kappa^2\Omega}{a^2+\kappa^2}\dot{x} = 0 \end{cases}$$

From these equations the authors deduced the differential equations

$$\begin{cases} \ddot{x} + (\dfrac{\kappa^2\Omega}{a^2+\kappa^2})^2 x = C_4 \\ \ddot{y} + (\dfrac{\kappa^2\Omega}{a^2+\kappa^2})^2\, y = C_5 \end{cases}$$

where C_4 and C_5 are arbitrary constants. Clearly, all theses relations can be obtained from the A-model by putting $\beta = 0$.

In [NF] the authors affirm that from these equations follow, in general case, that the sphere on a rough rotating horizontal plane describes an ellipses whose parameters depend on the initial conditions and which does not move in the fixed system of coordinates.

Now we compare the obtained result deduced from the A-model for homogenous sphere on a horizontal plane rotating with a constant angular velocity Ω and the result obtained in [NF].

By compare the two mathematical model we observe that:

1. If $\Omega \neq 0$ then in the A-model the sphere describe the curve

$$\begin{cases} x = r(t)\sin \displaystyle\int (\frac{K_1 + 2C_3}{r^2(t)} - \frac{K_2 r^2(t)}{4})dt \\ y = r(t)\cos \displaystyle\int (\frac{K_1 + 2C_3}{r^2(t)} - \frac{K_2 r^2(t)}{4})dt \end{cases}$$

where $r(t)$ is a function of time determine from (4.29). In the equations proposed by [NF] the sphere describe the circumferences

$$(x - [(\frac{\kappa^2\Omega}{a^2+\kappa^2})^2]^{-1}C_3)^2 + (y - [(\frac{\kappa^2\Omega}{a^2+\kappa^2})^2]^{-1}C_4)^2 = C$$

where C_3, C_4, C are arbitrary constants.

2. In the A-mechanics the curve describe by the sphere depend of the values of the ω_z, which is not observe in the result given in [NF].

3. For the case when $\Omega = 0$ in the A-model the sphere describe the curve

$$\begin{cases} x = r(t)\sin\int(\frac{-a^2C_1+2C_3}{r^2(t)})dt \\ y = r(t)\cos\int((\frac{-a^2C_1+2C_3}{r^2(t)})dt \end{cases}$$

where $r(t)$ is the function of time determine from the elliptic integral

$$\int \frac{dr^2}{\sqrt{r^4 - \frac{4(-a^2C_1+C_2)}{(-a^2C_1)^2}r^2 - \frac{4C_3^2}{(-a^2C_1)^2}}} = \imath(t-t_0), C_1 \neq 0$$

In [NF] the sphere describe the straight line

$$C_5x - C_4y = C_0$$

where C_5, C_4, C_0 are arbitrary constants, which not depend of the values of ω_z.

The problem of rolling of the homogenous sphere on the horizontal plane has been considered by many authors, for example, Coriolis [Cor], Hemming [Hem], Routh [Routh], Singe [Singe], and others. The assumption that the sphere and the plane make contact at a point leads to the conclusion that the vertical component of the angular velocity remains constant in the inertial motion of the sphere and that the rotation of the sphere about the vertical axis does not influence the trajectory of the motion of the center of the sphere. This phenomena is not observe in the A-mechanic, where the vertical component remains constant if and only if

$$\beta\Omega(x\dot{x} + y\dot{y}) = 0,$$

and the influence of the rotation of the sphere about the vertical axis is considerable.

It is interesting to compare the obtained result from the A-model in the problem of rolling of the homogenous sphere on the horizontal plane which the results given in [Neimark] when they study the Rolling of a sphere on an absolutely rough surface rotating with a constant angular velocity Ω. In particular for ω_z the following equation is deduced

$$\dot{\omega}_z = \vartheta_1\omega_x - \vartheta_2\omega_y \qquad (4.30)$$

where ϑ_1, $\vartheta_2\,\vartheta_3$ are the component of the instantaneous angular velocity of the moving coordinates axes $O_1\xi\eta\zeta$ with the origin at the center of the sphere.

If we denote by ϱ_1, $\varrho_2$the radii of curvature of the normal section of the surface along the direction$O_1\xi$ and $O_1\eta$. Then

$$\theta_1 = -\frac{\dot{y}}{\varrho_2}, \quad \theta_2 = \frac{\dot{x}}{\varrho_1}$$

By inserting into (4.30) the conditions (4.27) we finally deduced the equation

$$a\dot{\omega}_z = \dot{x}\dot{y}(\frac{1}{\varrho_2} - \frac{1}{\varrho_1}) + \Omega(\frac{x\dot{x}}{\varrho_1} + \frac{y\dot{y}}{\varrho_2})$$

If we consider that

$$\varrho_1 = \varrho_2 = \frac{\kappa^2}{a\beta}$$

we deduced the equation with respect ω_z obtained from the A-model (see for instance formula (4.28)).

Clearly, the rolling of a sphere on a plane we obtain when $\varrho_1 = \varrho_2 = \infty$, i.e., when $\beta = 0$, and if $\varrho_1 = \varrho_2 = C = const$, then we obtain the rolling of a heavy sphere on a spherical surface.

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