INVERSE APPROACH
IN THE STUDY OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

We extend the Eruguin result exposed in the paper "Construction of the whole set of ordinary differential equations with a given integral curve" published in 1952 and construct a differential system in R^N which admits a given set of the partial integrals, in particular we study the case when theses functions are polynomials. We construct a non-Darboux integrable planar polynomial system of degree n with one invariant irreducible algebraic curve $g(x,y)=0$. For this system we analyze the Darboux integrability, Poincare's problem and 16th's Hilbert problem for algebraic limit cycles. We propose the upper bound for the maximum degree of the invariant curve and for the maximum numbers of the algebraic limit cycles.

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1. Introduction

Nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and, in general, in applied sciences.

By definition a real autonomous differential system is a differential system of the form

$$\dot{\mathbf{x}} = \mathbf{v}(x), \quad \mathbf{x} \in \mathrm{R}^N$$

where the dependent variables $x = (x^1, x^2, ..., x^N)$ are real , the independent variable (time t) is real and functions $\mathbf{v}(x) = (v^1(x), ..., v^N(x))$ are continuous functions in $\mathcal{D} \subset \mathrm{R}^N$.

Definition 1.1 *The smooth function g and the relation $g(x) = 0$ are said partial integral and invariant relation of the vector field $\mathbf{v}(x)$ respectively if*

$$dg(\mathbf{v})|_{g(x)=0} = 0.$$

In this paper we are mainly interested in to study the differential system which possess a given set of invariant relations.

It is always helpful to look at this problem from another point of view. In this paper, we take an alternative viewpoint of starting with a given set of invariant relations and determining the form of the system which has such a set as invariant set. [Gal1, Gal2, Chr, Llib, Koo, Baut, Dol, Darb, Jou, Sad1, Sad2, Ram1, Ram2].

This approach was first developed by Eruguin in the paper "Construction of the whole set of ordinary differential equations with a given integral curve" published in 1952 [Eru]. In that article the author stated and solved the problem of constructing a planar vector field for which the given curve is its invariant. It is important to observe that Eruguin considered only one curve, moreover he didn't require that this curve was necessarily algebraic.

Eruguin proved that the most general planar vector field $\mathbf{v}$ for which the given curve

$$g(x,y)=0 \tag{1.1}$$

is its invariant curve generates the following differential equations

$$\begin{cases} \dot{x}=\nu(x,y)\{g,\,x\}+a(x,y)\\ \dot{y}=\nu(x,y)\{g,\,y\}+b(x,y)\end{cases} \tag{1.2}$$

where and ν, a, b are arbitrary functions:

$$\begin{aligned} &\{g,\,f\}\equiv\partial_x g\partial_y f-\partial_y g\partial_x f\\ &dg(\mathbf{v})=\Phi(x,y),\quad \Phi|_{g=0}=0\end{aligned} \tag{1.3}$$

These Eruguin ideas were applied in different areas. In particular Zubov in [Zub] constructed the planar system with a given region of stability.

Zubov constructed the following vector field

$$\begin{cases} \dot{x}=f\gamma\{g_1,\,x\}+g_1(\gamma\{f,\,x\}+g_1\varphi d_1)\\ \dot{y}=f\gamma\{g_1,\,y\}+g_1(\gamma\{f,\,y\}+g_1\varphi d_2)\end{cases} \tag{1.4}$$

where f, γ, d_1, d_2, φ are arbitrary functions which he choose in such a away that

$$d_1(g_1\{f,\,y\}-f\{g_1,\,y\})+d_2(g_1\{f,\,x\}-f\{g_1,\,x\})=1 \tag{1.5}$$

Under this condition Zubov proved that the following relations holds

$$\begin{cases} dg_1(\mathbf{v})=g_1(\gamma\{f,\,g_1\}+g_1\varphi(d_1\{g_1,\,y\}-d_2\{g_1,\,x\}))\\ dG_1(\mathbf{v})=\varphi G_1,\quad G_1\equiv\exp\dfrac{h}{g_1}\end{cases} \tag{1.6}$$

Galliulin in [Gal2] determines the most general vector field in R^N for which the given relations

$$g_j(t,x^1,\,x^2,...,x^N)=0,\quad j=1,2,..,S<\,N$$

are the invariant relations, where $g_1,g_2,..,g_S$ are smooth independent functions. The constructed system is the following

$$\dot{\mathbf{x}} = \frac{1}{\Gamma}\sum_{i,j=1}^{S}\Gamma_{ij}(\Phi_j - \partial_t g_j)\,\mathbf{grad}\,g_i + \mathbf{Y}$$

where $\mathbf{x} = col(x^1, x^2, ..., x^N)$, $\mathbf{Y}$ is an arbitrary vector orthogonal to the vectors

$$\mathbf{grad}\,g_j = col(\partial_1 g_j, \partial_2 g_j, ..., \partial_N g_j), \quad j = 1, \ldots, S, \quad \partial_k \equiv \frac{\partial}{\partial x^k},$$

Γ is the Grama determinant, Γ_{ij} are the minors of Γ and $\Phi_1, ...\Phi_S$ are arbitrary functions:

$$\Phi_j|_{g_j=0} = 0, \quad j = 1, 2, .., S.$$

The aim of this paper is to extend the Eruguin-Galliulin ideas to the case when the number of the given invariant relations is bigger than $N-1$. The results which we expose have been systematically developed in [Sad1].

2. Definitions and statement of the main results

In this section we constructed the most general stationary differential system from the given set of partial integrals.

First of all we introduce the following concept and notations which we shall use below.

Definition 2.1 *We call the vector field:*

$$\mathbf{v} = -\frac{det}{\Upsilon}\begin{pmatrix} dg_1(\partial_1) & dg_1(\partial_2) & \ldots & dg_1(\partial_N) & \Phi_1 \\ dg_2(\partial_1) & dg_2(\partial_2) & \ldots & dg_2(\partial_N) & \Phi_2 \\ \vdots & \ldots & \vdots & \vdots & \\ dg_M(\partial_1) & dg_M(\partial_2) & \ldots & dg_M(\partial_N) & \Phi_M \\ dg_{M+1}(\partial_1) & dg_{M+1}(\partial_2) & \ldots & dg_{M+1}(\partial_N) & \lambda_{M+1} \\ \vdots & \ldots & \vdots & \vdots & \\ dg_N(\partial_1) & dg_N(\partial_2) & \ldots & dg_N(\partial_N) & \lambda_N \\ \partial_1 & \partial_2 & \ldots & \partial_N & 0 \end{pmatrix} \tag{2.1}$$

the Eruguin-Galliulin Vector Fields, where $g_1,\, g_2,\, ...,\, g_N$ are smooth functions, $\Phi_1, \Phi_2, ..., \Phi_M$ are the Eruguin functions and $\lambda_{M+1}, \lambda_{M+2}, ..., \lambda_N$ are arbitrary functions:

$$\begin{cases} dg_k(\mathbf{v}) = \Phi_k, \quad \Phi_k|_{g_k=0} = 0, \quad k = 1, .., M, \\ dg_j(\mathbf{v}) = \lambda_j, \quad j = M+1, .., N, \end{cases} \tag{2.2}$$

$dg_1, dg_2, ..., dg_M$ are given independents 1-forms and $dg_{M+1}, dg_{M+2}, ..., dg_N$ are arbitrary 1-forms which we choose in such a way that

$$\Upsilon \equiv det\begin{pmatrix} dg_1(\partial_1) & dg_1(\partial_2) & \ldots & dg_1(\partial_N) \\ dg_2(\partial_1) & dg_2(\partial_2) & \ldots & dg_2(\partial_N) \\ \vdots & \ldots & \ldots & \vdots \\ dg_N(\partial_1) & dg_N(\partial_2) & \ldots & dg_N(\partial_N) \end{pmatrix} \equiv \{g_1,\, g_2,\, ..., g_N\} \neq 0. \tag{2.3}$$

The functions $\Phi_1,\, \Phi_2, ..., \Phi_M$ *we call the Eruguin functions* [Gal2].

We can identify the vector field (2.1) with the first order differential system

$$\dot{\mathbf{x}} = \Upsilon M^{-1}\mathbf{w}, \tag{2.4}$$

where M and $\mathbf{w}$ are the matrices:

$$\mathcal{M} = \Big(\big(dg_j(\partial_k)\big)_{j,k=1,2,..,N}\Big),$$
$$\mathbf{w} = col(\Phi_1,\, ..., \Phi_M,\, \lambda_{M+1}, ..., \lambda_N).$$

It is easy to show that the system (2.4) admits the equivalent representation

$$\dot{x}^j = \Phi_1\{x^j, g_2, ..g_{M+1}..., g_N\} + \ldots + \Phi_M\{g_1, .., x^j,\, g_{M+1}, .., g_N\} + Y^j, \tag{2.5}$$

where

$$Y^j = \lambda_{M+1}\{g_1, ..., g_M,\, x^j,\, g_{M+2}, .., g_N\} + \lambda_N\{g_1, ..., g_M, g_{M+1}, ..., g_{N-1},\, x^j\},$$
$$j = 1, 2, .., N.$$

Clearly, the vector $\mathbf{Y} = col(Y^1,\, Y^2, ..., Y^N)$ is orthogonal to the vectors $\mathbf{grad} g_j,\ j = 1, 2, .., M$, hence we obtain the Galliulin result [Gal2].

Example 2.1 We shall construct the Eruguin-Galliulin vector field for the case when the arbitrary functions

$$g_{M+1},\, g_{M+2},\, ..., g_{M+K}, \quad N = M + K$$

are such that

$$\begin{cases} dg_{M+1}(\mathbf{v}) = & Lg_{M+1} + L_1 g \\ dg_{M+2}(\mathbf{v}) = & Lg_{M+2} + L_1 g_{M+1} + L_2 g \\ & \vdots \\ dg_{M+K}(\mathbf{v}) = & Lg_{M+K} + L_1 g_{M+K-1} + ... + L_K g \\ dg(\mathbf{v}) = & L\, g, \\ g = & \prod_{j=1}^{M} g_j^{\tau_j}, \quad \tau_j \in \mathrm{C}, \end{cases} \tag{2.6}$$

where $L_1,\, L_2, ..., L_K,\, L$ are arbitrary functions.

By introducing the functions $G_1,\, G_2,\, ..., G_K$:

$$g_{M+j} = G_j g, \quad j = 1, 2, .., K,$$

we obtain

$$\begin{cases} dG_1(\mathbf{v}) =& L_1 \\ dG_2(\mathbf{v}) =& L_1G_1 + L_2 \\ & \vdots \\ dG_K(\mathbf{v}) =& L_1G_{K-1} + ... + L_k \end{cases}$$

Clearly, the arbitrary functions $\lambda_{M+1},\, \lambda_{M+2}, ..., \lambda_N = \lambda_{M+K}$ in this case we determine as follow

$$\begin{cases} \lambda_{M+1} =& g(LG_1 + L_1) \\ \lambda_{M+2} =& g(LG_2 + L_1G_1 + L_2) \\ & \vdots \\ \lambda_{M+K} =& g(LG_K + L_1G_{K-1} + ... + L_K) \end{cases}$$

Let us introduce the 1-forms $\omega_1,\, \omega_2, ..., \omega_K$:

$$\begin{cases} dG_1 =& \omega_1 \\ dG_2 =& G_1\omega_1 + \omega_2 \\ & \vdots \\ dG_K =& G_{K-1}\omega_1 + ... + \omega_K. \end{cases}$$

After some straightforward calculations we prove that

$$\begin{cases} \omega_j = d\Upsilon_j, \\ \omega_j(\mathbf{v}) = L_j, \quad j = 1, 2, .., K. \end{cases}$$

We determine the functions $\Upsilon_1,\, \Upsilon_2,\, ...\Upsilon_K$ as follows

$$d\Theta = \Psi^{-1} dG,$$

where

$$d\Theta = col(d\Upsilon_1,\, d\Upsilon_2,\, ..., d\Upsilon_K),$$
$$dG = col(dG_1,\, dG_2, ...dG_K),$$
$$\Psi = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ G_1 & 1 & 0 & 0 & \dots & 0 \\ G_2 & G_1 & 1 & 0 & \dots & 0 \\ G_3 & G_2 & G_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & 0 \\ G_K & G_{K-1} & G_{K-2} & \dots & G_1 & 1 \end{pmatrix}.$$

In particular for $\Upsilon_1,\, \Upsilon_2, ..., \Upsilon_5$ we have

$$\begin{cases} \Upsilon_1 = G_1 \\ \Upsilon_2 = G_2 - \dfrac{G_1^2}{2} \\ \Upsilon_3 = G_3 - G_1G_2 + \dfrac{G_1^3}{3} \\ \Upsilon_4 = G_4 - G_1G_3 + G_1^2G_2 - \dfrac{G_1^4}{4} - \dfrac{G_2^2}{2} \\ \Upsilon_5 = G_5 - G_1G_4 + G_1^2G_3 - G_1^3G_2 + \dfrac{G_1^5}{5!} + \dfrac{G_2^3}{3!}. \end{cases}$$

For the function Υ_j there are the equivalent representations

$$\Upsilon_j = \sum_{m=1}^{K-1} \alpha_{jm}(x) g^{m-j}$$

where $\alpha = (\alpha_{jm})$ is some matrix.

Corollary 2.1

Let us suppose that the functions

$$L,\ L_1,\ ...,\ L_K$$

are such that

$$\sum_{j=0}^{K} \nu_j L_j = 0, \quad L_0 = L,$$

where $\nu_0,\ \nu_1, ..., \nu_K$ are constants, then the constructed system (2.5),(2.6) admits the first integral

$$F(x) = g^{\nu_0} \exp \sum_{j=1}^{K} \nu_j \Upsilon_j = \prod_{j=1}^{K} g_j^{\nu_0 \tau_j} \exp \sum_{j=1}^{K} \nu_j \Upsilon_j,$$

For the particular case when

$$\Phi_j = K_j(x)\, g_j$$

then the condition on the existence the first integral F takes the form

$$\nu_0 \sum_{j=1}^{M} \tau_j K_j + \sum_{j=1}^{K} \nu_j L_j = 0,$$

For the planar polynomial vector field with invariant algebraic curves $g_j = 0$, $j = 1, ..., M$ this condition was deduced in [Chr et al.1]

Proposition 2.1 *Let*

$$g_j(x) = 0, \quad x = (x^1, x^2, ..., x^N), \quad j = 1, 2, .., M < N$$

are invariant relations of a differential system (S).

Assume that

$$\Upsilon = \{g_1,\, g_2,\, ...,\, g_M, g_{M+1} ..., g_N\} \neq 0,$$

for arbitrary smooth functions $g_{M+1},\, g_{M+2}, .., g_N$.

Then the following statement hold:

System $(\mathcal{S})$ *can be written as (2.5).*

Proof. Suppose that

$$\dot{x} = X(x) \tag{2.7}$$

is a differential system having $g_1,\, g_2, ..., g_M$ as partial integrals. Then taking

$$\begin{cases} \Phi_j = \dfrac{1}{\Upsilon}(\mathbf{grad}\, g_j,\, X), \quad j = 1, 2, .., M \\ \lambda_j = \dfrac{1}{\Upsilon}(\mathbf{grad}\, g_j,\, X), \quad j = M + 1, 2, .., N \end{cases}$$

we get that the system (2.4), or, what is the same, (2.5) becomes system (2.7). Note that in the definition of Φ_j and λ_j we have used that $\{g_1,\, g_2,\, ...,\, g_N\} \neq 0$.

Now we shall study the case when the given number of partial integrals is $S > N$.

If $S = N$ then the differential system (2.5) takes the form

$$\dot{x}^j = \Phi_1\{x^j ..., g_M, ..., g_N\} + \ldots + \Phi_N\{g_1, ..., g_M, .., x^j\} \quad j = 1.., N. \tag{2.8}$$

Proposition 2.2 *The differential system (2.8) admits the complementary invariant relation*

$$g_\nu(x) = 0, \quad \nu = N + 1, ..., S$$

if and only if

$$\det \begin{pmatrix} dg_1(\partial_1) & dg_1(\partial_2) & \ldots & dg_1(\partial_N) & \Phi_1 \\ dg_2(\partial_1) & dg_2(\partial_2) & \ldots & dg_2(\partial_N) & \Phi_2 \\ \vdots & \ldots & \vdots & \vdots & \\ dg_M(\partial_1) & dg_M(\partial_2) & \ldots & dg_M(\partial_N) & \Phi_M \\ dg_{M+1}(\partial_1) & dg_{M+1}(\partial_2) & \ldots & dg_{M+1}(\partial_N) & \Phi_{M+1} \\ \vdots & \ldots & \vdots & \vdots & \\ dg_N(\partial_1) & dg_N(\partial_2) & \ldots & dg_N(\partial_N) & \Phi_N \\ dg(\partial_1) & dg(\partial_2) & \ldots & dg(\partial_N) & \Phi_\nu \end{pmatrix} = 0, \tag{2.9}$$

or, what is the same,

$$\Phi_1\{g,...,g_M,...,g_N\}+...+\Phi_N\{g_1,...,g_M,...,g\}+\Phi_\nu\{g_1,...,g_M,...,g_N\}=0. \tag{2.10}$$

We obtain the proof from the equality

$$dg_\nu(\mathbf{v})=\Phi_\nu,$$

which in view of (2.1) coincides with (2.9).

Below we shall use the following identity

$$\begin{gathered}\{f_1,f_2,...,f_{N-1},g_1\}\{g_2,g_3,...,g_N,G\}+\\ +\{f_1,f_2,...,f_{N-1},g_2\}\{g_1,g_3,...,g_N,G\}+...\\ +\{f_1,f_2,...,f_{N-1},g_N\}\{g_1,g_2,...,g_{N-1},G\}+\\ \{f_1,f_2,...,f_{N-1},G\}\{g_1,g_2,...,g_{N-1},g_N\}\equiv 0.\end{gathered} \tag{2.11}$$

Its proof follow from by considering that (2.11) is equivalent to the relation

$$det\begin{pmatrix} dg_1(\partial_1) & dg_1(\partial_2) & \dots & dg_1(\partial_N) & \{f_1,f_2,f_3,...,f_{N-1},g_1\}\\ dg_2(\partial_1) & dg_2(\partial_2) & \dots & dg_2(\partial_N) & \{f_1,f_2,f_3,...,f_{N-1},g_2\}\\ \vdots & \dots & \vdots & \vdots & \\ dg_N(\partial_1) & dg_N(\partial_2) & \dots & dg_N(\partial_N) & \{f_1,f_2,f_3,...,f_{N-1},g_N\}\\ dG(\partial_1) & dG(\partial_2) & \dots & dG(\partial_N) & \{f_1,f_2,f_3,...,f_{N-1},G\}\end{pmatrix}\equiv 0 \tag{2.12}$$

It is easy to show that the Eruguin functions Φ_k, determined by the formula

$$\begin{gathered}\Phi_k=\sum_{\alpha_1,\,\alpha_2,\,..,\alpha_{N-1}=1}^{S+N}\lambda^1_{\alpha_1}\lambda^2_{\alpha_2}...\lambda^{N-1}_{\alpha_{N-1}}\{g_{\alpha_1},\,g_{\alpha_2}...,g_{\alpha_{N-1}},\,g_k\}G_{\alpha_1,\alpha_2...\alpha_{N-1}}(x),\\ k=1,2,..,S\end{gathered} \tag{2.13}$$

are the most general solutions of (2.10), where $G_{\alpha_1,\alpha_2...\alpha_{N-1}}$, $\lambda^1_{\alpha_1},\,\lambda^2_{\alpha_2},...\lambda^{N-1}_{\alpha_{N-1}}$ are arbitrary continuous functions:

$$\begin{cases} G_{\alpha_1,\alpha_2...\alpha_{N-1}}(x)|_{g_{\alpha_1}=.....=g_{\alpha_{N-1}}=0}\neq 0,\\ \Phi_k|_{g_k=0}=0, \quad k=1,2,....,S.\end{cases}$$

and $g_{S+j}=x_j$. The differential system (2.8) in this case takes the form

$$\dot{x}^j=\sum_{\alpha_1,\,\alpha_2,\,..,\alpha_{N-1}=1}^{S+N}\tilde{\lambda}^1_{\alpha_1}\tilde{\lambda}^2_{\alpha_2}...\tilde{\lambda}^{N-1}_{\alpha_{N-1}}\{g_{\alpha_1},\,g_{\alpha_2}...,g_{\alpha_{N-1}},\,x^j\}G_{\alpha_1,\alpha_2...\alpha_{N-1}}(x) \tag{2.14}$$

where $\tilde{\lambda}^k_{\alpha_j}=\{g_1,g_2,g_3,...,g_N\}\lambda^k_{\alpha_j}$.

Proposition 2.3 *Let $g_1(x), g_2(x), ..., g_S(x) \quad S > N$ are partial integrals of a differential system ($\mathcal{S}$).*

Assume that

$\Upsilon = \{g_1,\, g_2, ..., g_N\} \neq 0,$

then the following statement hold:

System ($\mathcal{S}$) can be written as (2.14).

The proof is analogously to the proof of the proposition 2.2.

In particular when

$$G\alpha_1, \alpha_2 ... \alpha_{N-1}(x) = \prod_{\substack{m=1 \\ (m \neq \alpha_j, j=1,...,N-1)}}^{S} g_m \tag{2.15}$$

then the differential equations (2.14) take the form:

$$\dot{x}^j = \sum_{\alpha_1,\, \alpha_2,\, ..,\alpha_{N-1}=1}^{S+N} \tilde{\lambda}^1_{\alpha_1} \tilde{\lambda}^2_{\alpha_2} ... \tilde{\lambda}^{N-1}_{\alpha_{N-1}} \{g_{\alpha_1},\, g_{\alpha_2} ..., g_{\alpha_{N-1}},\, x^j\} \prod_{\substack{m=1 \\ (m \neq \alpha_j, j=1,...,N-1)}}^{S} g_m \tag{2.16}$$

As usual we denote by $\mathrm{R}[x]$ the ring of all real polynomials in the variables $x \equiv (x_1,\, x_2, ,,,,\, x_N)$. We consider the polynomial vector field in R^N, with degree n, i.e.,

$$\begin{cases} \mathbf{v} = (v^1(x), ..., v^N(x)), \quad v^1(x)\,, v^2(x), ..., v^N(x) \in \mathrm{R}[x], \\ n = max(deg(v^1(x)), ..., deg(v^N(x))) \end{cases}$$

Definition 2.2 *We say that $\{g = 0\} \subset \mathrm{R}^N$ is an invariant algebraic hypersurface of the polynomial vector field $\mathbf{v}$ of degree n if there exists a polynomial $K \in \mathrm{R}[x]$ such that*

$$dg(\mathbf{v}) = K(x)\, g.$$

The polynomial K at the degree at most $n-1$ is called the cofactor of $g(x) = 0$.

Definition 2.3 *A nonconstant (multivalued) function is said to be Darboux if it is of the form*

$$f = \ln(\prod_{j=1}^{S} g_j^{\sigma_j}(x)), \quad ,$$

where $\sigma_j \in \mathrm{C}$, $j = 1, 2, .., S$ are certain constants.

Definition 2.4 *We shall say that the vector field (2.8) with invariant relations*

$$g_j(x) = 0, \quad j = 1, ..., S > N, \tag{2.17}$$

is integrable if it admits $N-1$ independent first integrals $f_1,\ f_2,\ ..,f_{N-1}$, and integrable in the Darboux sense if

$$g_1,\ g_2, \ldots.., g_S$$

are polynomial functions and $f_1,\ f_2, ..., f_{N-1}$ are Darboux functions.

Darboux proved the following theorem.

Darboux's theorem

Si l'on connait $\frac{m(m+1)(m+2)....(m+n-1)}{n!} = M_n$ intégreles particuliéres algébriques de systéme

$$\frac{dx_1}{L_1} = \frac{dx_2}{L_2} = ... = \frac{dx_n}{L_n}, \quad L_1,\ L_2,\ ..., L_n \in \mathrm{R}[x]$$

on pourra trouver lemultiplicateur du systéme.

Si l'on connait M_n+r intégrales particuliéres algébriques du méme systéme, on pourra en determiner le multiplicateur et r intégrales générales.

Si l'on connait $M_n + n - 1 = q$ intégrales particuliéres algébriques $u_1,\ u_2,\ ..., u_q$ on pourra effectuer l'intégration compléte. Les intégrales se présenteront sous la forma suivante:

$$\begin{cases} u_1^{\alpha_1} u_2^{\alpha_2} ... u_q^{\alpha_q} = C_1 \\ u_1^{\beta_1} u_2^{\beta_2} ... u_q^{\beta_q} = C_2 \\ \qquad \vdots \\ u_1^{\lambda_1} u_2^{\lambda_2} ... u_q^{\lambda_q} = C_{n-1}. \end{cases}$$

In [Jou] the following result is proved.

Jounolous Theorem

Let $\mathbf{v}$ be a polynomial vector field defined in C^N of degree $n > 0$. Then $\mathbf{v}$ admits $\frac{(n+N-1)!}{(n-1)!} + n$ irreducible invariant algebraic hipersurface if and only if $\mathbf{v}$ has a rational first integral.

Proposition 2.3 *The vector field (2.16) with invariant relations (2.17) is integrable if and only if the vector field $\mathbf{v}$ is such that*

$$dg_k(\mathbf{v}) = \tilde{\lambda}\{f_1,\ f_2,\ ..,f_{N-1}, g_k\}, \quad k = 1, 2, .., S \tag{2.18}$$

where $\tilde{\lambda}$ is an arbitrary function.

Proof. Let us suppose that the vector field $\mathbf{v}$ is integrable, then it admits the representation [Sad1, Ram3]

$$\mathbf{v} = \tilde{\lambda}\, det \begin{pmatrix} df_1(\partial_1) & df_1(\partial_2) & \ldots & df_1(\partial_N) \\ df_2(\partial_1) & df_2(\partial_2) & \ldots & df_2(\partial_N) \\ \vdots & \ldots & \vdots & \vdots \\ df_{N-1}(\partial_1) & df_{N-1}(\partial_2) & \ldots & df_{N-1}(\partial_N) \\ \partial_1 & \partial_2 & \ldots & \partial_N \end{pmatrix} \equiv \tilde{\lambda}\,\{f_1, ..., f_{N-1}, *\},$$

where $\tilde{\lambda}$ is an arbitrary function. Hence we obtain that

$$dg_j(\mathbf{v}) = \tilde{\lambda}\{f_1, ..., f_{N-1}, g_j\}$$

on the other hand from (2.2) we obtain that

$$dg_j(\mathbf{v}) = \Phi_j.$$

By compare both we deduce (2.18).

We obtain the reciprocity result as follows.

Let us suppose that (2.18) holds. Clearly that the condition (2.10) holds identically in this case.

By inserting (2.18) in (2.8) we obtain

$$\begin{aligned}\mathbf{v}(*) = \lambda(\{f_1, f_2, f_3, ..., f_{N-1}, g_1\}\{*, g_2, g_3, ..., g_N\}+ \\ ...+\{f_1, f_2, f_3, ..., f_{N-1}, g_N\}\{g_1, g_2, g_3, ..., *\}).\end{aligned}$$

In view of the identity (2.11) we deduce

$$\mathbf{v}(*) = \{g_1, g_2, g_3, ..., g_N\}\lambda\{f_1, f_2, f_3, ..., f_{N-1}, *\} \equiv \tilde{\lambda}\{f_1, ..., f_{N-1}, *\}.$$

as a consequence the vector field is integrable.

Corollary 2.2 *Let us suppose that the system (2.16) is polynomial of degree n. Then it is Darboux integrable if and only if*

$$\begin{cases} \tilde{\lambda}^{\alpha}_j = \sigma^{\alpha}_j = constants, \quad j = 1, 2, .., S, \; \alpha = 1, 2, .., N-1 \\ \tilde{\lambda}^{\alpha}_{S+1} = \nu_{\alpha}\{f^{\alpha}, x^2, x^3, ..., x^N\}, ..., \; \tilde{\lambda}^{\alpha}_{S+N} = \nu_{\alpha}\{x^1, x^2, x^3, ..., f^{\alpha}\} \end{cases} \tag{2.19}$$

where $\nu_1, \nu_2, ..., \nu_{N-1}$ are arbitrary rational functions.

The proof follows from the fact that in view of (2.19) we obtain

$$\begin{cases} dg_k(\mathbf{v}) = g\Big(1 + \sum_{j=1}^{N-1} \nu_j + \sum_{j,k=1,\, j\neq k}^{N-1} \nu_j\nu_k + ... + \prod_{j=1}^{N-1} \nu_j\Big)\{f_1, ..., f_{N-1}, g_k\} \\ k = 1, 2, .., S. \end{cases}$$

Example 2.2 We shall suppose that the given invariant relations of the differential system (2.16) are the hyperplane

$$x^j = 0, \quad j = 1, 2, .., N.$$

We choose the Eruguin functions as follows

$$\Phi_j = \Psi_j(x^j)\frac{\{\varphi_1, ..., \varphi_{N-1}, x^j\}}{\{\varphi_1, \varphi_2 ..., \varphi_N\}},$$

where $\varphi_{kj}(x^k)$, $\Psi_j(0)=0$, $k,j=1,2,..,N$ and φ_j, $j=1,2,..,N$ are arbitrary functions. Hence we obtain that this system takes the form

$$\dot{x}^j = \Psi_j(x^j)\frac{\{\varphi_1, ..., \varphi_{N-1},\, x^j\}}{\{\varphi_1, \varphi_2 ..., \varphi_N\}}, \quad j=1,2,..,N \tag{2.20}$$

We shall study the case when

$$\begin{cases} \{\varphi_1, \varphi_2 ..., \varphi_N\} \neq 0 \\ d\varphi_j = \sum_{k=1}^{N} \varphi_{kj}(x^k)dx^k, \quad j=1,2,..,N \end{cases} \tag{2.21}$$

The differential system (2.20), (2.21) is integrable.

In fact, by considering that

$$\sum_{k=1}^{N} \frac{\varphi_{kj}(x^k)dx^k}{\Psi_k(x^k)} = \begin{cases} 1, \text{if } j=N \\ 0, \text{if } j\neq N \end{cases}$$

we deduce the existence of $N-1$ independents first integrals

$$f_j(x) \equiv \sum_{k=1}^{N} \int \frac{\varphi_{kj}(x^k)}{\Psi_k(x^k)} dx^k = c_j, \quad j=1,2,...,N-1. \tag{2.22}$$

It is easy to show that the vector field $\mathbf{v}(*)$ is such that

$$\mathbf{v}(*) = g\,\frac{\{f_1, ..., f_{N-1},\, *\}}{\{\varphi_1, \varphi_2 ..., \varphi_N\}}, \quad g = \prod_{k=1}^{N} \Psi_k(x^k).$$

For the subcase when the invariant hyperplane are such that

$$x^k - a_{m+k} = 0, \quad k=1,2,..,N,\; m=1,2,...,M$$

and

$$\begin{cases} \varphi_{kj}(x^k) = (x^k)^{j-1}, \quad k,j=1,2,..,N \\ \Psi_k(x^k) = \prod_{m=1}^{M}(x^k - a_{m+k}), \end{cases} \tag{2.23}$$

then the first integrals (2.22) in this case take the form

$$\begin{cases} f_j = \ln \prod_{k=1}^{N}\prod_{m=1}^{M}(x^k - a_{m+k})^{\sigma^j_{k+m}}, \quad j=1,..,N \\ \sigma^j_{m+k} = \dfrac{(a_{k+m})^j}{\prod_{l=1,\, l\neq m}^{M}(a_{k+l} - a_{k+m})} \end{cases}$$

as a consequence the system (2.20),(2.21),(2.23) is Darboux integrable.

An interesting particular case is the following

$$\dot{x}^j = x^j \prod_{m=1}^{M} (\frac{x_j^2}{m^2} - 1) \frac{\{\varphi_1, ..., \varphi_{N-1},\, x^j\}}{\{\varphi_1, \varphi_2 ..., \varphi_N\}}. \tag{2.24}$$

Hence, by making $M \to +\infty$ we deduce

$$\dot{x}^j = x^j \prod_{m=1}^{+\infty} (\frac{x_j^2}{m^2} - 1) \frac{\{\varphi_2, ..., \varphi_N,\, x^j\}}{\{\varphi_1, \varphi_2 ..., \varphi_N\}} \equiv \sin \pi x^j \frac{\{\varphi_2, ..., \varphi_N,\, x^j\}}{\{\varphi_1, \varphi_2 ..., \varphi_N\}}. \tag{2.25}$$

We observe that the differential systems of the type (2.20) appear in the theory of the Stäckel mechanical system [Sad1, Ram3]. With respect to this system we state the following problem:

Determine the real constants K_1, K_2, ..., K_{N-1}, L in such a way that the hyperplane

$$x^N = \sum_{j=1}^{N-1} K_j x^j + L$$

is invariant of the system (2.20).

We solve this problem for the case when

$$N = 2, \quad \varphi_2 = \frac{a}{2}(x^2 + y^2) + xy,$$
$$\Psi_1(x) = -\lambda \prod_{j=1}^{M} (\frac{x^2}{a_j^2} - 1), \quad \Psi_2(y) = \lambda \prod_{j=1}^{M} (\frac{y^2}{a_j^2} - 1)$$

where $a, a_1, ..., a_M$ are real constants and λ is an arbitrary function.

The system (2.20) in this case takes the form

$$\begin{cases} \dot{x} = \lambda\,(x + ay) \prod_{j=1}^{M} (\frac{x^2}{a_j^2} - 1) \\ \dot{y} = \lambda\,(ax + y) \prod_{j=1}^{M} (\frac{y^2}{a_j^2} - 1) \end{cases} \tag{2.26}$$

We require to determine the real values of the constants K and L in such a way

$$y = Kx + L \tag{2.27}$$

is an invariant straight line of (2.26).

Clearly that the parameter K must be satisfies the relation

$$K^{2M} + aK^{2M-1} - aK - 1 = 0.$$

Hence we obtain that $K_1 = 1$, $K_2 = -1$ satisfies this relation. For $M > 2$ there exist at most four real values of K which satisfy this equation.

By using the algebraic computer packages we can solve the stated problem. In particular for the cubic and quintic system when

$$\lambda = (\prod_{j=1}^{M} a_j^2)^{-1},$$
$$a = \sqrt{5}, \quad a_1 = 1, \quad M = 1$$
$$a = \sqrt{5}, \quad a_1 = 1, \quad a_2 = \sqrt{5} - 2, \quad M = 2$$

we obtain respectively

$$K_1 = 1, \quad L_1 = 0 \quad K_2 = -1, \quad L_2 = 0$$

$$\begin{cases} K_1 = 1, \quad L_1 = 0, \quad K_2 = -1, \quad L_2 = 0, \\ K_3 = -\frac{1}{2} - \frac{\sqrt{5}}{2}, \quad L_3 = -\frac{1}{2} + \frac{\sqrt{5}}{2} \\ K_4 = \frac{1}{2} - \frac{\sqrt{5}}{2}, \quad L_4 = -\frac{3}{2} + \frac{\sqrt{5}}{2}, \\ K_3 = -1/2 - \frac{\sqrt{5}}{2}, \quad L_5 = 1/2 - \frac{\sqrt{5}}{2}, \\ K_4 = 1/2 - \frac{\sqrt{5}}{2}, \quad L_6 = \frac{3}{2} - \frac{\sqrt{5}}{2} \end{cases}$$

The quintic polynomial system in this case was constructed in [Art et al.]

$$\begin{cases} \dot{x} = (x + \sqrt{5}y)(x^2 - 1)(x^2 - (\sqrt{5} - 2)^2) \\ \dot{y} = (\sqrt{5}x + y)(y^2 - 1)(y^2 - (\sqrt{5} - 2)^2) \end{cases}$$

and admits 14 straight lines.

The Eruguin functions in this case are:

$$\Phi_m = (x + \sqrt{5}\, y)(x^2 - 1)(x^2 - (\sqrt{5} - 2)^2),$$
$$\Phi_{4+m} = (\sqrt{5}\, x + y)(y^2 - 1)(y^2 - (\sqrt{5} - 2)^2), \quad m = 1, 2, 3, 4$$

hence the constructed system is not Darboux integrable.

For the case when in (2.26)

$$a_j = j, \quad a = 0$$

we obtain the system

$$\begin{cases} \dot{x} = x \prod_{j=1}^{M} (\frac{x^2}{j^2} - 1) \\ \dot{y} = y \prod_{j=1}^{M} (\frac{y^2}{j^2} - 1) \end{cases} \tag{2.28}$$

In this case the system admits the following invariant straight line

$$\begin{aligned}&x=j, \quad x=-j, \quad j=1,2,..,M\\&y=j, \quad y=-j\\&y=x, \quad y=-x\end{aligned}$$

Clearly, the system (2.28) is Darboux integrable

If $M=+\infty$ then the system (2.28) takes the form

$$\begin{cases}\dot{x}=x\prod_{j=1}^{\infty}(\frac{x^2}{j^2}-1)=\sin\pi x\\ \dot{y}=y\prod_{j=1}^{\infty}(\frac{y^2}{j^2}-1)=\sin\pi y\end{cases}$$

for which the infinity numbers of the straight lines

$$\begin{aligned}&x=j, x=-j, \quad j=1,2,..,+\infty\\&y=j, \quad y=-j \quad j=1,2,..,+\infty\\&y=x+2m, \quad y=-x+2m, \quad m\in \mathrm{Z}\end{aligned}$$

are its invariant.

The problem of the determination of the upper bound for the maximum number of the invariant straight lines ($L(n)$) for the polynomial system is an open problem.

It is easy to show that [Ram2]

$$L(n)\geq\begin{cases}2n+1, & \text{when } n \text{ is even,}\\ 2n+2, & \text{when } n \text{ is odd.}\end{cases}$$

There exist the following conjecture

Conjecture [Art et al.]

$$L(n)\leq 3n-1$$

This upper bound is reached in particular for $n=2,3,4,5$.

3. Inverse approach for the planar vector fields

In this section we analyze the Eruguin-Galliulin theory developed in the above section for the case when $N=2$.

The differential system (2.8) in this case take the form

$$\begin{cases}\dot{x}=\Phi_1\{x,g_2\}+\Phi_2\{g_1,x\}=P(x,y)\\ \dot{y}=\Phi_1\{y,g_2\}+\Phi_2\{g_1,y\}=Q(x,y),\end{cases}, \tag{3.1}$$

we set $\mathbf{v}=(P,Q)$.

For the case when this equations admit the subsidiary invariant curves

$$g_j(x, y) = 0, \quad j = 3, 4, ..., S,$$

The Eruguin functions must be satisfy the relations (see formula (2.10))

$$\Phi_1\{g_j, g_2\} + \Phi_2\{g_1, g_j\} + \Phi_j\{g_2, g_1\} = 0 \quad j = 1, 2, ...S. \tag{3.2}$$

Hence the Eruguin functions Φ_m can be determine as follows

$$\begin{cases} \Phi_m = \sum_{j=1}^{S} \tilde{\lambda}_j(x)\{g_j,\, g_m\} \prod_{k=1,\, k\neq j}^{S} g_k + (\tilde{\lambda}_{S+1}\{x,\, g_m\} + \tilde{\lambda}_{S+2}\{\, g_m,\, y\})g, \\ g = \prod_{j=1}^{S} g_j \end{cases} \tag{3.3}$$

where $\tilde{\lambda}_1,\, \tilde{\lambda}_2, ..., \tilde{\lambda}_{S+2}$, are arbitrary functions.

The veracity of this representation we obtain by inserting (3.3) into (3.2) and by considering the identity (2.11) which in this case takes the form

$$\{g_k,\, g_j\}\{g_i,\, g_m\} + \{g_k,\, g_m\}\{g_j,\, g_i\} + \{g_i,\, g_k\}\{g_j,\, g_m\} \equiv 0, \tag{3.4}$$

Corollary 3.1 *The differential equations (3.1), (3.3) can be rewritten as follows*

$$\begin{cases} \dot{x} = g(x, y)(\sum_{j=1}^{S} \tilde{\lambda}_j(x, y)\dfrac{\{g_j, x\}}{g_j} + \tilde{\lambda}_{S+2}) = P(x, y) \\ \dot{y} = g(x, y)(\sum_{j=1}^{S} \tilde{\lambda}_j(x, y)\dfrac{\{g_j, y\}}{g_j} - \tilde{\lambda}_{S+1}) = Q(x, y) \end{cases} \tag{3.5}$$

We deduce the proof by inserting (3.3) into (3.1) by using the identity (3.4).

From these relations we obtain the following proposition

Proposition 3.1 *Let $g_j(x, y) = 0$, $j = 1, 2, ..., S$ are the irreducible algebraic curves, then the polynomial differential system (3.5) admits the Darboux first integral*

$$f(x, y) = \ln(\prod_{j=1}^{S} g_j^{\sigma_j})$$

if and only if in (3.3)

$$\tilde{\lambda}_j = \nu_0\sigma_j = constants, \quad \tilde{\lambda}_{S+1} = \nu\{f, y\}, \quad \tilde{\lambda}_{S+2} = \nu\{x, f, \},$$

where ν_0, ν are arbitrary rational functions and $\sigma_j = constants$, $j = 1, 2, ..., S$.

The proof is easy to obtain from (3.5).

The system (3.5) in this case takes the form

$$\begin{cases} \dot{x} = V\{x,\, F\} \\ \dot{y} = V\{y,\, F\} \end{cases}$$

where

$$F = \prod_{j=1}^{S} g_j^{\sigma_j}, \quad V = g(\frac{\nu_0}{F} + \nu).$$

We illustrate the above results in the following concrete cases.

Example 3.1

In this section we give the results exposed in [Sad1], related with the construction the planar polynomial vector field with invariant circumferences:

$$g_j(x,y) \equiv (x-a_j)^2 + (y-b_j)^2 - r_j^2 = 0, \quad j = 1,2,...,S$$

The system (3.5) under the restrictions

$$\tilde{\lambda}_{S+1} = \tilde{\lambda}_{S+2} = 0$$

takes the form

$$\begin{cases} \dot{x} = -\sum_{j=1}^{S} \tilde{\lambda}_j (y-b_j) \prod_{m=1,\, m\neq j}^{S} g_m \equiv P(x,y) \\ \dot{y} = \sum_{j=1}^{S} \tilde{\lambda}_j (x-a_j) \prod_{m=1,\, m\neq j}^{S} g_m \equiv Q(x,y), \end{cases} \tag{3.6}$$

or, what is the same,

$$\begin{cases} \dot{x} = -y(x^2+y^2)^{S-1} \sum_{j=1}^{S} \tilde{\lambda}_j - (x^2+y^2)^{S-1} \sum_{j=1}^{S} \tilde{\lambda}_j b_j + .. \equiv P(x,y) \\ \dot{y} = x(x^2+y^2)^{S-1} \sum_{j=1}^{S} \tilde{\lambda}_j + (x^2+y^2)^{S-1} \sum_{j=1}^{S} \tilde{\lambda}_j a_j + ... \equiv Q(x,y). \end{cases}$$

Now we determine the arbitrary functions $\lambda_1,\, \lambda_2,\, ..., \lambda_S$ in such a way that the above vector field is polynomial of the fixed degree n.

Corollary 3.2 *Let us suppose that*

$$\begin{cases} \tilde{\lambda}_j \in \mathrm{R}[x,y], \quad j = 1,2,..,S \\ S_1 = deg \sum_{j=1}^{S} \tilde{\lambda}_j, \\ S_2 = max(deg(\sum_{j=1}^{S} \tilde{\lambda}_j b_j), deg(\sum_{j=1}^{S} \tilde{\lambda}_j a_j) \end{cases}$$

then

$$n = max(deg(P), deg(Q)) \leq 2S - 1 + S_1,$$

if

$$\sum_{j=1}^{S} \tilde{\lambda}_j \neq 0,$$

and

$$n = max(deg(P), deg(Q)) \leq 2S - 2 + S_2,$$

if

$$\sum_{j=1}^{S} \tilde{\lambda}_j = 0.$$

From this results we obtain the proof of the following result

Proposition 3.2 *Every configuration of the circumferences in the plane is realizable by a polynomial of the degree at most $2S + S_1 - 1$ or $2S + S_2 - 2$ where S_j, $j = 1, 2$ are the degree of the polynomials introduced above.*

In a paper [Llib4] the authors proved that every configuration of cycles on the plane is realizable (up to homeomorphism) by a polynomial of the degree at most $2(m + r) - 1$, where m is the number of cycles and r is the number of primary cycles (a cycle C is primary if there are no other cycles contained in the bounded region limited by C).

It is interesting to observe that the upper bound for the degree of the constructed vector field is independent from whether its cycles are primary or not.

Now we shall study the case when the circumferences form two nests with the centers at the points $(0, 0)$ and $(a, 0)$ respectively, hence

$$\begin{aligned} g_j(x, y) &\equiv x^2 + y^2 - r_j^2 = 0, \quad j = 1, 2, ..., l_1, \\ g_j(x, y) &\equiv (x - a)^2 + y^2 - r_j^2 = 0, \quad j = l_1, ..., S \end{aligned}$$

Clearly, for this case the Eruguin functions are such that

$$\begin{cases} \Phi_1 = \Phi_3 = ... = \Phi_{l_1} = 4ay \displaystyle\sum_{j=l_1+1}^{S} \tilde{\lambda}_j \prod_{m=1,\, m\neq 2}^{l_1} g_m \\ \Phi_2 \equiv \Phi_{l_1+1} = \Phi_{l_1+2} = ... = \Phi_S = -4ay \displaystyle\sum_{j=1}^{l_1} \tilde{\lambda}_j g_2 \prod_{m=l_1+1}^{S} g_m. \end{cases}$$

The differential system (3.6) takes the form

$$\begin{cases} \dot{x} = -2(\displaystyle\sum_{j=1}^{l_1} \tilde{\lambda}_j \prod_{k=l_1+1}^{S} g_k + \sum_{j=l_1+1}^{S} \tilde{\lambda}_j \prod_{k=l}^{l_1} g_k) y, \\ \dot{y} = 2(\displaystyle\sum_{j=1}^{l_1} \tilde{\lambda}_j \prod_{k=l_1+1}^{S} g_k + \sum_{j=l_1+1}^{S} \tilde{\lambda}_j \prod_{k=l}^{l_1} g_k) x - 2a \sum_{j=l_1+1}^{S} \tilde{\lambda}_j \prod_{k=l}^{l_1} g_k. \end{cases}$$

In particular for the case when

$$\begin{cases} n = 2l + 1 = S + 1, \quad l_1 = l, \\ -2\sum_{j=1}^{l} \tilde{\lambda}_j = x - a + y, \\ -2\sum_{j=l+1}^{S} \tilde{\lambda}_j = x + y, \end{cases}$$

we obtain the vector field constructed in [Sad1].

By designating by $F_a(x,y)$, $F_0(x,y)$ the following polynomials

$$\begin{cases} F_a(x,y) = (x + y - a)\prod_{j=2}^{l+1}((x-a)^2 + y^2 - r_j^2) \quad l \geq 1, \\ F_0(x,y) = F_a(x,y)|_{a=0}. \end{cases}$$

we can deduce that the above vector field takes the form:

$$\begin{cases} \dot{x} = \Big(F_0(x,y) - F_a(x,y)\Big)\, y = P(x,y) \\ \dot{y} = -\Big(F_0(x,y) - F_a(x,y)\Big)\, x + aF_0(x,y) = Q(x,y). \end{cases}$$

This system has the following properties:

1) has only 3 critical points in the finite plane R^2

$$(0,0), \quad (\frac{a}{2}, 0), \quad (a,\, 0).$$

2) the Liapunov quantities σ and Δ for the system are :

i)

$$\begin{cases} \sigma(0,0) = \sigma(a,0) \\ \Delta(0,0) = \Delta(a,0). \end{cases} .$$

ii)

$$\begin{cases} \sigma(0,0) = (-1)^l a \prod_{j=1}^{l} r_j^2 \\ \Delta(0,0) = a^2 \prod_{j=1}^{l} l(a^2 - r_j^2)\Big(\prod_{j=1}^{l}(a^2 - r_j^2) - (-1)^l \prod_{j=1}^{l} r_j^2\Big) \\ \sigma^2(0,0) - 4\Delta(0,0) = a^2\Big(\big(2\prod_{j=1}^{l}(a^2 - r_j^2) - (-1)^l \prod_{j=1}^{l} r_j^2\big)^2 - 8\big(\prod_{j=1}^{l}(a^2 - r_j^2)\big)^2\Big). \end{cases} ,$$

$$
\begin{cases}
\sigma(\frac{a}{2},0) = a\prod\limits_{j=1}^{l}\left((\frac{a}{2})^2 - r_j^2\right) \\
\Delta(\frac{a}{2},0) = -\frac{a^4}{2}\prod\limits_{j=1}^{l}\left((\frac{a}{2})^2 - r_j^2\right)\sum\limits_{l=1}^{l}\prod\limits_{j=1,\, j\neq l}^{l}\Big((\frac{a}{2})^2 - r_j^2\Big).
\end{cases}.
$$

The circumferences do not intersect if

$$
r_j < a/2, \quad j = 1, ..., l,
$$

so

$$
\Delta(\frac{a}{2},0) < 0,
$$

and, as a consequence the critical point $(\frac{a}{2},0)$ is a saddle.

It is evident that the other critical points are the stability or non stability foci depending on whether k is odd or even.

Hence we obtain that the constructed polynomial vector field of degree $n = S+1$ admits $S = 2l$ invariant circumferences.

The proposition 3.1 we illustrate in the next two examples.

Example 3.2 The particular case of the Lienard equation

$$
\ddot{x} - \frac{d}{dx}h(x)\dot{x} - \alpha h(x)\frac{d}{dx}h(x) = 0
$$

or, what is the same,

$$
\begin{cases}
\dot{x} = y + h(x) \\
\dot{y} = \alpha h(x)\frac{d}{dx}h(x)
\end{cases},
$$

is Darboux integrable.

In fact, the first integral F in this case is the following

$$
F(x,y) = g_1^{\sigma_1} g_2^{\sigma_2}
$$

where g_1, g_2, σ_1, σ_2 are such that

$$
\begin{cases}
g_1(x,y) = y + \frac{1+\sqrt{4\alpha+1}}{2}h(x) \\
g_2(x,y) = y + \frac{1-\sqrt{4\alpha+1}}{2}h(x) \\
\sigma_1 = -\frac{1-\sqrt{4\alpha+1}}{2\sqrt{4\alpha+1}}, \quad \sigma_2 = \frac{1+\sqrt{4\alpha+1}}{2\sqrt{4\alpha+1}}
\end{cases}
$$

It is easy to show that in this case the Eruguin functions are

$$
\Phi_1 = \frac{1+\sqrt{4\alpha+1}}{2\sqrt{4\alpha+1}}g_1, \quad \Phi_2 = \frac{1-\sqrt{4\alpha+1}}{2\sqrt{4\alpha+1}}g_2
$$

Clearly, if $4\alpha+1<\ 0$ then the function F takes the form

$$F(x,y)=(y^2+h(x)y-\alpha h^2(x))\exp\left(\sqrt{-\alpha-\frac{1}{4}}arctan\frac{\sqrt{-\alpha-\frac{1}{4}}h(x)}{y+\frac{1}{2}h(x)}\right)$$

It is interesting to observe that if the function h admits the following development

$$h(x)=x+a_2x^2+...$$

then the origin of the given system is a focus.

Example 3.3 The differential equation

$$\dot{z}=i\,(a_{10}z+a_{01}\bar{z}+\sum_{j+k=3}a_{jk}z^j\,\bar{z}^k) \tag{3.7}$$

is Darboux integrable, where $a=(a_{jk}),\ j,k=0,2,3$ are real constants matrix and

$$z=x+iy,\quad \bar{z}=x-i\,y$$

are the complex coordinate in the plane R^2.

In fact, the equations (3.7) are equivalent to the cubic planar system

$$\begin{cases}\dot{x}=y(a_{01}-a_{10}+(a_{12}-a_{21}+3(a_{03}-a_{30})\,x^2+(a_{12}-a_{21}+a_{30}-a_{03})\,y^2)\\ \dot{y}=x(a_{01}+a_{10}+(a_{12}+a_{21}+a_{03}+a_{30})\,x^2+(a_{12}+a_{21}-3(a_{30}+a_{03})y^2).\end{cases}$$

Hence, by introducing the correspondent notations we obtain the system

$$\begin{cases}\dot{x}=y(a+b\,x^2+c\,y^2)\\ \dot{y}=x(\alpha+\beta\,x^2+\gamma\,y^2).\end{cases} \tag{3.8}$$

We shall analyze the case when $c\neq 0$.

Let $g_1,\ g_2$ are the functions:

$$g_j(x,y)=\nu_j(x^2-\lambda_0)-y^2+\lambda_1,\quad j=1,2$$

where $\lambda_0,\ \lambda_2,\ \nu_1,\ \nu_2$ are constants:

$$\begin{cases}\lambda_0=\dfrac{\gamma\,a-\alpha\,c}{b\gamma-c\beta},\quad \lambda_1=\dfrac{\alpha\,b-\beta\,a}{b\gamma-c\beta},\\ \nu_1=\dfrac{\gamma-b}{2c}+\sqrt{(\dfrac{\gamma-b}{2c})^2+\dfrac{\beta}{c}},\quad \nu_2=\dfrac{\gamma-b}{2c}-\sqrt{(\dfrac{\gamma-b}{2c})^2+\dfrac{\beta}{c}}\\ \nu_1-\nu_2\neq 0,\end{cases}$$

then the following relations hold

$$\begin{cases} dg_j(\mathbf{v}) = 2x\,y\,(\gamma - \nu_j\,c)g_j, \quad j = 1,2 \\ \{g_1,\, g_2\} = 4x\,y\,(\nu_1 - \nu_2). \end{cases} \tag{3.9}$$

The proof is easy to obtain after some calculations.

The given vector field is Darboux integrable with F :

$$F(x,y) = \frac{(\nu_1(x^2 - \lambda_0) - y^2 + \lambda_1)^{b+\nu_1 c}}{(\nu_2(x^2 - \lambda_0) - y^2 + \lambda_1)^{b+\nu_2 c}},$$

here we use the relation

$$c(\nu_1 + \nu_2) = \gamma - b.$$

Now we shall study the case when ν_1, ν_2 are complex numbers.

By introducing the notations

$$\gamma - b = 2cq, \quad \gamma + b = 2cr$$

we obtain that

$$\nu_1 = q + i\,p, \quad \nu_2 = q - i\,p, \quad p^2 = -4\beta\,c - q^2,\, p\, > 0$$

The system (3.8) takes then the form (we put $c = 1$)

$$\begin{cases} \dot{x} = y(a + (r - q)\,x^2 + y^2) \\ \dot{y} = x(\alpha - (p^2 + q^2)\,x^2 + (r + q)y^2). \end{cases} \tag{3.10}$$

By considering that in this case

$$g_1(x,y) = q(x^2 - \lambda_0) - y^2 + \lambda_1 + ip(x^2 - \lambda_0),$$

we obtain that the first integral F takes the form

$$F(x,y) = \Big((y^2 - \lambda_1 - q(x^2 - \lambda_0))^2 + p^2(x^2 - \lambda_0)^2\Big) exp\Big(2r\, arctg\frac{p(x^2 - \lambda_0)}{y^2 - \lambda_1 - q(x^2 - \lambda_0)}\Big).$$

4. Planar differential system with one invariant algebraic curve

In this section, by applying the results of the section 3, we construct the analytic planar vector field

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y), \end{cases} \tag{4.1}$$

where P and Q are analytic functions on the region $G \subset \mathrm{R}^2$, from a given set of trajectories:

$$\begin{cases} g_j(x,y) = y - y_j(x) = 0, \quad j = 1, 2, \ldots, S \geq 2 \\ \{g_1,\, g_2\} = y_2^{'}(x) - y_1^{'}(x) \neq 0, \\ \quad \displaystyle\prod_{j=1}^{S} y_j(x) \neq 0 \end{cases} \tag{4.2}$$

where $\frac{dy_j}{dx} = y_j^{'}$ and $y_1,\, y_2,\, \ldots, y_S \in C^r(G \subset \mathrm{R}),\, r \geq 1$.

We shall study the particular case when $y_1,\, y_2,\, ...y_s$ are solutions of the equation

$$g(x, y_j) = 0, \quad j = 1, 2, .., S$$

where $g(x, y) = 0$ is an algebraic irreducible curve.

By considering (2.8) we obtain that the require system (4.1), (4.2) :

$$\begin{cases} \dot{x} = \Phi_1(x, y) - \Phi_2(x, y) \\ \dot{y} = \Phi_1(x, y)y_2^{'}(x) - \Phi_2(x, y)y_1^{'}(x) \end{cases}. \tag{4.3}$$

The condition (3.2), on the existence of the complementary partial integrals, in this case take the form

$$\Phi_1(y_2(x) - y_j(x)))' + \Phi_2(y_j(x) - y_1(x))' + \Phi_j(y_1(x) - y_2(x))' = 0, \quad j = 3, 4 \ldots, S. \tag{4.4}$$

The given set of differential equations (3.5) in this case can be rewritten as follows

$$\begin{cases} \dot{x} = \sum_{j=1}^{S} \tilde{\lambda}_j \prod_{m \neq j} (y - y_m) + \tilde{\lambda}_{S+2} g = P(x, y) \\ \dot{y} = \sum_{j=1}^{S} \tilde{\lambda}_j y_j^{'} \prod_{m \neq j} (y - y_m) - \tilde{\lambda}_{S+1} g = Q(x, y) \end{cases} \tag{4.5}$$

or, what is the same,

$$\begin{cases} \dot{x} = y^S \tilde{\lambda}_{S+2} + y^{S-1}(\sum_{j=1}^{S} \tilde{\lambda}_j - \tilde{\lambda}_{S+2} \sum_{j=1}^{S} y_j) + y^{S-2}(\sum_{j=1}^{S} \tilde{\lambda}_j y_j - \sum_{j=1}^{S} y_j \sum_{k=1}^{S} \tilde{\lambda}_k) + ..) + ... \\ \dot{y} = -y^S \tilde{\lambda}_{S+1} + y^{S-1}(\sum_{j=1}^{S} \tilde{\lambda}_j y_j^{'} - \tilde{\lambda}_{S+1} \sum_{j=1}^{S} y_j) + y^{S-2}(\sum_{j=1}^{S} \tilde{\lambda}_j y_j^{'} - \sum_{j=1}^{S} y_j \sum_{k=1}^{S} \tilde{\lambda}_k y_k y_k^{'} ...) + ... \end{cases} . \tag{4.6}$$

we set $\mathbf{v} = (P, Q)$.

Proposition 4.1 *Let us suppose that the arbitrary functions* $\tilde{\lambda}_1,\, ...\tilde{\lambda}_S,\, \tilde{\lambda}_{S+1},\, \tilde{\lambda}_{S+2}$ *are such that*

$$\begin{cases} \tilde{\lambda}_{S+1} = -q_0(x), \quad \tilde{\lambda}_{S+2} = p_0(x) \\ \tilde{\lambda}_j = \frac{\triangle_j}{\triangle_0}(\sum_{k=0}^{S} p_k(x) y_j^{S-k}(x)) \\ \tilde{\lambda}_j y_j^{'} = \frac{\triangle_j}{\triangle_0}(\sum_{k=0}^{S} q_k(x) y_j^{S-k}(x)), \end{cases} \tag{4.7}$$

where p_k, q_k are continuous functions on $D \subset \mathrm{R}$.

$$\triangle_0 = det \begin{pmatrix} 1 & 1 & \dots & 1 & \vdots & 1 \\ y_1 & y_2 & \dots & y_j & \vdots & y_S \\ \vdots & \vdots & \dots & \vdots & \vdots & \\ y_1^{S-1} & y_2^{S-1} & \dots & y_j^{S-1} & \vdots & y_S^{S-1} \end{pmatrix},$$
$$\triangle_j = det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & \vdots & 1 \\ y_1 & y_2 & \dots & y_{j-1} & y_{j+1} & \vdots & y_S \\ \vdots & \vdots & \dots & \vdots & \vdots & & \\ y_1^{S-2} & y_2^{S-2} & \dots & y_{j-1}^{S-2} & y_{j+1}^{S-2} & \vdots & y_S^{S-2} \end{pmatrix}, \tag{4.8}$$

Then the differential system (4.6) takes the form

$$\begin{cases} \dot{x} = p_0(x)y^S + ... + p_S(x) = P(x,y) \\ \dot{y} = q_0(x)y^S + ... + q_S(x) = Q(x,y) \end{cases} \tag{4.9}$$

and we set $\mathbf{v} = (P, Q)$.

We shall study the case when p_0, p_1, ..., p_S, q_0, q_1, ...q_S are polynomials on the variable x and such that $\mathbf{v}$ represented a polynomial vector field of degree

$$n = max(degP,\, degQ)$$

Corollary 4.1 *Let g be a irreducible polynomial on the variables x and y:*

$$g = a_0(x)\prod_{j=1}^{S}(y - y_j(x)) = \sum_{j=0}^{S} a_j(x)y^{S-j}, \tag{4.10}$$

where

$$\begin{cases} a_1 = -\,a_0(x)\sum_{j=1}^{S} y_j(x) \\ a_2 = a_0(x)\prod_{j<k} y_j(x)y_k(x) \\ \quad\vdots \\ a_S = (-1)^S a_0(x)\prod_{j=1}^{S} y_j(x) \end{cases}$$

then

$$dg(\mathbf{v}) = (\sum_{j=1}^{S}\frac{\Phi_j}{g_j})g = K(x)g$$

where K is the cofactor of $g = 0$.

Proposition 4.2 *Let the curve (4.10) is invariant of the non zero polynomial system of degree* n :

$$\begin{cases} \dot{x} = r_1(x)y^{n-1} + ... + r_n(x) = P(x,y) \\ \dot{y} = q_0(x)y^n + ... + q_n(x) = Q(x,y) \end{cases} \tag{4.11}$$

where $r_j(x)$, $q_j(x)$, $j = 0, 1, .., n$ *are polynomials of degree* j *in the variable* x, *then*

$$S \leq 2n.$$

Proof, al absurd, let us suppose that $S = 2n + 1$, then from (4.9), (4.11) we obtain that

$$\begin{cases} p_{n+1}(x) = 0, \\ p_j(x) = 0, \\ q_j(x) = 0, \quad j = 0, 1, ..n, \end{cases}$$

on the other hands from (4.7), (4.8) we obtain that

$$\tilde{\lambda}_j(x) = 0, \quad j = 1, 2, ..., 2n + 2$$

hence the vector field is a zero vector field. Contradiction.

5. Quadratic system with one invariant algebraic curve

In this section we shall study the case in which the vector field (4.9) is quadratic i.e.,

$$\begin{cases} \dot{x} = p_{S-2}y^2 + p_{S-1}(x)y + p_S(x) = P(x,y) \\ \dot{y} = q_{S-2}y^2 + q_{S-1}(x)y + q_S(x) = Q(x,y) \end{cases}$$

where $max(degP, \, degQ) = 2$ and q_{S-j}, p_{S-j}, $j = 0, 1, 2$ are polynomials in the variable x. Below, for simplicity we shall denote this system as follows

$$\begin{cases} \dot{x} = p_0y^2 + p_1(x)y + p_2(x) = P(x,y) \\ \dot{y} = q_0y^2 + q_1(x)y + q_2(x) = Q(x,y) \end{cases} \tag{5.1}$$

First we prove the following general results related with the system (5.1).

Proposition 5.1 *Let us suppose that (5.1) is such that*

$$dg(\mathbf{v}) = (\alpha_0 y + \alpha x + \beta)g, \tag{5.2}$$

where g *is given in the formula (4.10) and* α_0, α, β *are real constants and* p_j, q_j *are polynomials of degree* j *in the variable* x :

$$\begin{cases} p_j = \sum_{k=0}^{j} p_{jk}x^k, \\ q_j = \sum_{k=0}^{j} q_{jk}x^k, \quad j = 0, 1, 2. \end{cases} \tag{5.3}$$

Then,

If $p_0 \neq 0$ *hence*

$$\begin{cases} \max(deg\, a_j(x)) \leq j, \\ \max(deg\, g) \leq S. \end{cases} \tag{5.4}$$

If

$$\begin{cases} p_0 = 0, \quad p_{11} \neq 0, \\ \alpha_0 = (Sk+m)p_{11}, \quad q_0 = kp_{11} \\ Sk+m \in \mathrm{N} \end{cases} \tag{5.5}$$

hence

$$\begin{cases} \max(deg\, a_j(x)) \leq kj+m, \quad j = 1, 2, .., S \\ \max(deg\, g) \leq Sk+m. \end{cases} \tag{5.6}$$

In fact, from (5.2) we deduced the differential system

$$\begin{cases} A \cdot \dfrac{d\mathbf{a}}{dx} = B \cdot \mathbf{a} \\ p_0 \dfrac{da_0}{dx} = 0 \\ p_2 \dfrac{da_S}{dx} + q_2 a_{S-1} = (\alpha x + \beta) a_S \end{cases} \tag{5.7}$$

where

$$\mathbf{a} = col(a_0,\, a_1,\, ..., a_S)$$

is a vector and A, B are matrix which we determine respectively as follow

$$\begin{pmatrix} p_1 & p_0 & 0 & 0 & 0 & \dots & 0 \\ p_2 & p_1 & p_0 & 0 & 0 & \dots & 0 \\ 0 & p_2 & p_1 & p_0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & p_2 & p_1 & p_0 \\ 0 & 0 & 0 & 0 & 0 & p_2 & p_1 \end{pmatrix},$$

$$\begin{pmatrix} \alpha_0 - Sq_0 & 0 & 0 & 0 & 0 & \dots \\ \alpha x + \beta - Sq_1 & \alpha_0 - (S-1)q_0 & 0 & 0 & 0 & \dots \\ -Sq_2 & \alpha x + \beta - (S-1)q_1 & \alpha_0 - (S-2)q_0 & 0 & 0 & \dots \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots \\ 0 & 0 & \dots & -2q_2 & \alpha x + \beta - q_1 & \alpha_0 \\ 0 & 0 & \dots & 0 & -q_2 & \alpha x + \beta \end{pmatrix},$$

From (5.7) we easily deduce that if $p_0 \neq 0$ then the coefficients $a_j, \quad j = 0, 1, 2, .., S$ are polynomials of degree at most j.

For the second case, after integration we easily deduce that

$$\mathbf{a} = \mathcal{R}\mathbf{p} \tag{5.8}$$

where

$$\mathbf{p} = col(p_1^m, \, p_1^{m+1}, ..., p_1^{m+k}, ..., p_1^{m+kS})$$

is a vector and $\mathcal{R}$ is the following constant matrix

$$\begin{pmatrix} R_0^0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ R_0^1 & R_1^1 & 0 & 0 & 0 & 0 & \dots & R_k^1 & 0 & \dots & 0 & \dots & \dots & 0 & 0 \\ R_0^2 & R_1^2 & R_2^2 & 0 & 0 & 0 & \dots & R_k^2 & R_{k+1}^2 & 0 & \dots & 0 & R_{2k}^2 & 0 & 0 \\ R_0^3 & R_1^3 & R_2^3 & R_3^3 & 0 & 0 & \dots & R_k^3 & R_{k+1}^3 & R_{k+2}^3 & 0 & 0 & R_{2k}^3 & R_{2k+1}^3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \dots & 0 & \dots & \dots & 0 \\ R_0^S & R_1^S & R_2^S & \dots & \dots & R_S^S & 0 & R_k^S & R_{k+1}^S & R_{k+2}^S & \dots & 0 & R_{2k}^S & \dots & R_{Sk}^S \end{pmatrix}$$

Hence we easily obtain the veracity of our assertion.

Corollary 5.1 *If the invariant algebraic curve is non reducible and* $k = \frac{q_0}{p_{11}} \geq 0$ *then* $m = 0$.

In fact, if $m \neq 0$ and under the indicated condition we have that the given invariant curve is reducible.

Proposition 5.2 *The maximum degree of the irreducible invariant algebraic curve of the non Darboux integrable quadratic system (5.1) is* 12. *if* $p_{11} \neq 0$

In fact from (5.8), (5.5) by considering that the given curve is irreducible, then we obtain that $m = 0$.

On the other hand from the last of equation of (5.7) in particular we deduce that if $k > 3$ then

$$\frac{1}{p_{11}^2} R_{Sk}^S (p_{10}^2 p_{22} - p_{10} p_{21} p_{11} + p_{20} p_{11}^2) = 0. \tag{5.9}$$

Hence, if $R_{Sk}^S = 0$ then the degree of the algebraic curve is not maximal. On the other hand if

$$p_{10}^2 p_{22} - p_{10} p_{21} p_{11} + p_{20} p_{11}^2 = 0$$

then the quadratic system is Darboux integrable.

Corollary 5.2 *Let us suppose that the algebraic curve*

$$g(x, y) = \sum_{l=0}^{S} \sum_{j=0}^{kS} \mathcal{R}_j^l (p_{11} x + p_{10})^{j+m} y^l = 0 \tag{5.10}$$

is invariant curve of the maximum degree of the quadratic vector field. Then:

$$\mathbf{v} = ((p_{11}x + p_{10})y + p_{22}x^2 + p_{21}x + p_{20})\partial_x + (3p_{11}y^2 + (q_{11}x + q_{10})y + q_{22}x^2 + q_{21}x + q_{20})\partial_y$$

and

$$dg(\mathbf{v}) = \frac{12}{p_{11}} (p_{11}^2 y + p_{22} x + p_{21} p_{11} - 2 p_{22} p_{10}) g.$$

This results can be extended analogously for the polynomial system of degree n.

Proposition 5.3 *Let us give the invariant curve with S branches of non-Darboux integrable polynomial system of degree n :*

$$\begin{cases} \dot{x} = \sum_{j=0}^{n} p_j(x) y^{n-j} = P(x,y) \\ \dot{y} = \sum_{j=0}^{n} q_j(x) y^{n-j} = Q(x,y) \end{cases} \tag{5.11}$$

with cofactor $K = \sum_{j=0}^{n-1} \alpha_j(x) y^{n-1-j}$ where $\alpha_j(x) = \sum_{k=0}^{j} \tau_{jk} x^k$, and $\tau = (\tau_{jk})$ is a constant real matrix.

Then

$$\max deg(g) \leq \begin{cases} S, & \text{if } p_0 \neq 0 \\ S(n+1), & \text{if } p_0 = 0,\ p_{11} \neq 0 \end{cases}$$

For the case when $p_{11} = 0$ it is easy to show that

$$\begin{cases} \alpha_0 = S q_0, \\ \max deg\, a_j \leq j\,, \\ \max deg\, g \leq S. \end{cases}$$

From this result and proposition 4.2 we deduce the proof of the following proposition.

Proposition 5.4 *The maximum degree of the invariant curve of the non Darboux integrable polynomial planar vector field of degree n ($\mathcal{N}(n)$) is $2n(n+1)$,* i.e.,

$$\mathcal{N}(n) \leq 2n(n+1). \tag{5.12}$$

6. Quadratic system with one invariant algebraic curve. Examples

First we study the case when $S = 2$, i.e., we analyze the quadratic system with invariant algebraic curve of the type

$$g(x,y) = a_0(x) y^2 + a_1(x) y + a_2(x) = 0 \tag{6.1}$$

where a_0, a_1, a_2 are polynomial on the variable x.

We shall study the cases when $p_0 = 0$, $\quad p_{11} \neq 0$ and $p_0 \neq 0$.

For the first case the he equation (5.10) takes the form

$$g(x,y) = \sum_{l=0}^{2} \sum_{j=0}^{2k} \mathcal{R}_j^l (p_{11} x + p_{10})^j y^l = 0. \tag{6.2}$$

Clearly, the maximum degree of this curve is six.

We shall illustrate this particular case in concrete examples.

Example 6.1. The quadratic vector field

$$\begin{cases} \dot{x} = ax^2 + (-2ac + y + b)x + 25C_1a^2 - 10aC_1d + C_1d^2 \\ \dot{y} = 3y^2 + (-12ac + 6b + dx)y + (-150C_1a^2C_3 - 3a^2 + 60C_1adC_3 + \\ \quad da - 6C_1C_3d^2)x^2 + (db - 2adc)x - 750a^3C_1^2C_33 + 225C_1a^3 + 450a^2C_1^2dC_3 \\ \quad - 2d^3C_1 - 140C_1a^2d + 12a^2c^2 + 3b^2 - 90d^2C_1^2aC_3 - 12cab + 29C_1d^2a + 6d^3C_3C_1^2 \end{cases}$$

where C_1, C_3 are arbitrary nonzero constants, admits the invariant curve of degree six

$$\begin{aligned} g =& C_1y^2 + (x^3 + (C_1d - 3C_1a)x - 4C_1ac + 2C_1b)y + x^6C_3 + (\frac{1}{2}d - \frac{3}{2}a)x^4 + (b - 2ac)x^3 \\ &+ (-3C_3C_1^2d^2 - 9aC_1d + 21C_1a^2 - 75C_3C_1^2a^2 + C_1d^2 + 30C_3C_1^2ad)x^2 + \\ &(-3C_1ab - 2C_1dac + C_1db + 6a^2C_1c)x + 150C_33C_1^3a^2d - \frac{75}{2}a^2C_1^2d + \\ &\frac{15}{2}d^2C_1^2a + C_11b^2 - 250C_3C_1^3a^3 - \frac{1}{2}d^3C_1^2 + 2C_3C_1^3d^3 + \frac{125}{2}a^3C_1^2 - \\ &4C_1abc - 30C_3C_1^3d^2a + 4a^2c^2C_1 = 0 \end{aligned}$$

with cofactor $6(y + ax - 2ac + b)$.

Example 6.2 Now we study the case in which $S = 2, \quad p_0 \neq 0$. Clearly, in this case the invariant algebraic curve of the quadratic system is the family of the conics

$$g = a_0y^2 + (a_{11}x + a_{10})y + a_{22}x^2 + a_{21}x + a_{20} = 0 \tag{6.5}$$

where a_0, a_{11}, a_{10}, a_{22}, a_{21}, a_{20} are real constants.

In particular, for the quadratic system

$$\begin{cases} \dot{x} = \beta\, y^2 - (\beta + 2)xy + (\beta - 4)x^2 - 2\beta x \\ \dot{y} = (\alpha - 4)y^2 - 2(\alpha + 2)xy + (\alpha - 4)x^2 - 2\alpha\, x \end{cases} \tag{6.7}$$

we have that $p_0 = \beta \neq 0$.

After some calculations we can prove that the invariant curve is the parabola

$$(y - x)^2 - 2x = 0$$

with the cofactor

$$K = 2(\beta + \alpha)y + 2(\beta + \alpha)x - 2\beta.$$

The quadratic system with invariant parabola was constructed in [Sad1] and admits the equivalent representation :

$$\begin{cases} \dot{x} = \beta\left((y - x)^2 - 2\,x\right) - 4\,x\,(x + y) \\ \dot{y} = \alpha\left((y - x)^2 - 2\,x\right) - 4\,(x + y)^2 \end{cases}. \tag{6.7}$$

The points

$$O(0,\,0),\quad N(\frac{1}{2},\,-\frac{1}{2}),\quad M(\frac{\beta^2}{K_3},\,\frac{\beta^2(2\alpha-\beta)}{K_3})$$

are its critical points, where $K_3\equiv 2((\alpha-\beta)^2-2\alpha)$. The bifurcation analysis show that this system is generic. The bifurcation curves divide the plane $(\alpha,\,\beta)$ in 17 region in which we observe a qualitative change in the behavior of the trajectories of the constructed quadratic system. We determine 38 different quadratic systems, among these there is one with one limit cycles [Rey].

The quadratic system

$$\begin{cases} \dot{x}=\dfrac{1}{2A}(Ax^2+By^2+2C)A_0+\dfrac{1}{A}(-2Byq_{11}-2xBq_{22}+xA_1A_2Bq_{21})y \\ \dot{y}=\dfrac{1}{2B}(A_1y^2+yq_{11}x+q_{22}x^2+q_{21}x+A_1C), \end{cases}$$

admits as invariant the curve

$$Ax^2+By^2+2C=0$$

with cofactor $K=A_0x+A_1y$, where $A,B,C,A_0,A_1,q_{11},q_{21},q_{22}$, are real constants such that $A\neq 0,\ B\neq 0$.

Now we study the case when $S=3$, i.e., we analyze the quadratic system with invariant algebraic curve of the type

$$g(x,y)=\sum_{l=0}^{2}\sum_{j=0}^{3k}\mathcal{R}_j^l(p_{11}x+p_{10})^jy^l=0. \tag{6.8}$$

Example 6.3 (The Filipstov system).
For the quadratic system

$$\begin{cases} \dot{x}=16(1+a)x-6(2+a)x^2+(2+12x)y \\ \dot{y}=3a(1+a)x^2+(15(1+a)-2(9+5a)x)y+16y^2 \end{cases}$$

we have that $3k=4\Longrightarrow m=0$.

As we can observe in this case the quadratic system possesses the irreducible invariant algebraic curve

$$g(x,y)\equiv y^3+\frac{1}{4}(3(1+a)-6(1+a)x)y^2+\frac{3}{4}(1+a)ax^2y+\frac{3}{4}(1+a)a^2x^4=0.$$

The cofactor of this curve is

$$K=48y-4(1+a)x+5(1+a).$$

Example 6.4 For the quadratic system

$$\begin{cases} \dot{x} = (2+3a)x^2 + (2+4y)x + y \\ \dot{y} = (5+4(1+a))y + 6y^2 + ax^2 \end{cases}$$

we obtain that $k = 2 \Longrightarrow m = 0$.

The quadratic system possesses the irreducible invariant algebraic curve of degree four [Chr et al.3]

$$a^2x^4 + +2ax^2(x+1)y + (1+x)y^2 + y^3 = 0$$

The cofactor K in this case is

$$K = 18y + (5+6a)x + 5.$$

For the case when $S = 4$ we obtain the invariant algebraic curve of the type

$$g(x,y) = \sum_{l=0}^{2}\sum_{j=0}^{4k} \mathcal{R}_j^l (p_{11}x + p_{10})^j y^l = 0. \tag{6.9}$$

Clearly, the maximum degree of this curve is 12. The upper bound is reached in particular in the following example [Chr et al.3]

Example 6.5 For the non Darboux integrable quadratic system

$$\begin{cases} \dot{x} = xy + x^2 + 1 \\ \dot{y} = 3y^2 - \dfrac{81}{2}x^2 + \dfrac{57}{2} \end{cases}$$

has we have that $k = 3 \Rightarrow m = 0$ as a consequence the above vector field has the invariant irreducible algebraic curve of the maximum degree 12. In the indicated paper was showed that the the curve

$$\begin{aligned} &-442368 - 7246584x^2 + 71546517x^4 - 97906500x^6 + 41343750x^8 - 23437500x^{10} + 48828125x^{12} \\ &+ (322272x - 12126312x^3 + 23463000x^5 + 1125000x^7 + 15625000x^9)y - \\ &(98784 - 711288x^2 + 5058000x^4 - 375000x^6)y^2 + (32928x - 1124000x^3)y^3 - 5488y^4 = 0 \end{aligned}$$

is its invariant.

7. Construction the polynomial planar system with invariant algebraic curves with variables separable

In this section we deal with the polynomial system with invariant algebraic curve with variables separable

$$g(x,y) = F_1(x) + F_2(y) = 0,$$

where F_1, F_2 are arbitrary polynomials :

$$deg(g(x,y)) = max\Big(deg(F_1(x)),\, max(deg(F_2(y))\Big).$$

We state and study the following problem.

Problem 7.1 Let g be a function:

$$g(x,y) = g_0 + A\int\prod_{j=1}^{m_1}(x-a_j)dx + B\int\prod_{j=1}^{m_2}(y-b_j)dy$$

where A, B, a_1, a_2, ..., a_{m_1}, b_1, b_2, ..., b_{m_2} are real parameters such that

$$a_1 < a_2 < .... < a_{m_1}, \quad b_1 < b_2..... < b_{m_2}, \quad AB \neq 0.$$

We require to determine the non-Darboux integrable polynomial vector $\mathbf{v}$ of degree n for which the given curve is its invariant.

We propose the solution of the state problem for the following particular cases

$$\begin{gathered} m_2 = n-1, m_1 = n-1, \\ m_2 = 2m-1, m_1 = 2m+1,\ n = 2m+1, \\ m_2 = 2m-2, m_1 = 2m,\ n = 2m, \\ m_2 = m_1 = m, \quad n = 2m+1, \\ m_2 = m_1 = m, \quad n = 2m+2. \end{gathered}$$

Proposition 7.1 The polynomial system of degree n [Sad2]

$$\begin{cases} \dot{x} = (Ax+By+C)\partial_y g(x,y) \equiv P(x,y) \\ \dot{y} = -(Ax+By+C)\partial_x g(x,y) + \lambda g(x,y) \equiv Q(x,y) \end{cases}$$

admits as invariant curve

$$g(x,y) = g_0 + K_1\int\prod_{j=1}^{n-1}(y-b_j)dy + K_2\int\prod_{j=1}^{n-1}(x-a_j)dx = 0,$$

where a_1, a_2, ..., a_{n-1}, b_1, b_2, ..., b_{n-1}, K_1, K_2, g_0, A, B, C, λ are arbitrary real parameters.

By choosing the arbitrary parameters properly we can construct the nonsingular algebraic curve of degree n, hence the genus ($\mathcal{G}$) of this curve is:

$$\mathcal{G} = \frac{1}{2}(n-1)(n-2)$$

Example 7.1 Let g is a nonsingular curve of degree $2m+2$ such that

$$g_m(x,y) = g_o + \int\prod_{j=1}^{m}x((\frac{x}{j\pi})^2-1)dx + \int\prod_{j=1}^{m}y((\frac{y}{j\pi})^2-1)dy = 0$$

It is easy to show that the polynomial system of degree $n = 2m+1$:

$$\begin{cases} \dot{x} = (A_m x + B_m y + C_m) \prod_{j=1}^{m} y((\frac{y}{j\pi})^2 - 1) \equiv P(x,y) \\ \dot{y} = -(A_m x + B_m y + C_m) \prod_{j=1}^{m} x((\frac{x}{j\pi})^2 - 1) + \lambda g_m(x,y) \equiv Q(x,y) \end{cases}$$

admits as invariant the given curve.

By considering that

$$\prod_{j=1}^{\infty}((\frac{y}{j\pi})^2 - 1) = \sin y \Rightarrow \lim_{m\to+\infty} g_m(x,y) = g_0 + \cos x + \cos y$$

and choose the arbitrary parameters $A_m,\ B_m,\ C_m$ properly, we obtain the analytic planar vector field

$$\begin{cases} \dot{x} = \sin y \equiv P(x,y) \\ \dot{y} = -\sin x + \lambda(g_0 + \cos x + \cos y) \equiv Q(x,y) \end{cases}$$

for which the curve

$$g_0 + \cos x + \cos\ y = 0$$

is its invariant.

Clearly, the constructed analytic system admits infinity many number of limit cycles.

Proposition 7.2 The polynomial vector field of degree n

$$\begin{cases} \dot{x} =(a + byx)\partial_y H(x,y) \\ \dot{y} =-\ (a + byx)\partial_x H(x,y) + (n+1)byH(x,y) \end{cases} \tag{7.1}$$

admits as invariant the algebraic curve of degree $n+1$

$$H(x,y) \equiv x^{n+1} + G_{n-1}(x,y) = 0, \tag{7.2}$$

where G_{n-1} is an arbitrary polynomial of degree $n-1$. Clearly, this system in general has no Darboux integrating factors or first integrals.

The following particular case is an interesting one:

Corollary 7.1 Let

$$g_m(x,y) = g_0 + \int_{x_0}^{x} \prod_{j=1}^{m+1} x(x^2 - a_j^2)dx + \int_{y_0}^{y} \prod_{j=1}^{m-1} y(y^2 - b_j^2)dy = 0$$

is a curve of degree $2m+2$ with the maximum genus $\mathcal{G} = 2(m+1)(m-1)+1$ is invariant of the vector field of degree $n = 2m+2$:

$$\begin{cases} \dot{x} =(a + b_m yx) \prod_{j=1}^{m-1} y(y^2 - b_j^2) \\ \dot{y} =-\ (a + b_m yx) \prod_{j=1}^{m+1} x(x^2 - a_j^2 + (2m+2)b_m y g_m(x,y) \end{cases}$$

Example 7.2 By making $m \to +\infty$ and choose the arbitrary parameters properly we deduce from the above system as a particular case the analytic system

$$\begin{cases} \dot{x} = & aJ_0(y) \\ \dot{y} = & -aJ_0(x) + \lambda y(J_1(x) + J_1(y) + g_0) \end{cases}$$

where $J_0,\ J_1$ are the Bessel functions. This analytic system admits an infinity many number of limit cycles.

Analogously we construct the polynomial system of degree $n = 2m$:

$$\begin{cases} \dot{x} = & (a + b_m yx)\prod_{j=1}^{m-1}(y^2 - b_j^2) \\ \dot{y} = & -(a + b_m yx)\prod_{j=1}^{m+1}(x^2 - a_j^2) + 2mb_m y g_m(x,y) \end{cases}$$

with invariant curve

$$g_m(x,y) = g_0 + \int_{x_0}^{x}\prod_{j=1}^{m+1}(x^2 - a_j^2)dx + \int_{y_0}^{y}\prod_{j=1}^{m-1}(y^2 - b_j^2)dy = 0$$

is a curve of degree $2m + 2$ with maximum genus $\mathcal{G} = 2m(m-1)$

By using the algebraic packages it is possible to show the following proposition.

Proposition 7.3 . There exist polynomials $p(x,y),\ q(x,y)$ of degree n for which the non-Darboux integrable differential system

$$\begin{cases} \dot{x} = & a\partial_y g(x,y) + p(x,y), \quad a = const. \\ \dot{y} = & -a\partial_x g(x,y) + q(x,y) \end{cases} \tag{7.3}$$

has the invariant curve of degree $n + 1$

$$g(x,y) = g_0 + A\int(\prod_{j=1}^{n}(x - a_j))dx + B\int(\prod_{j=1}^{n}(y - b_j))dy = 0, \tag{7.4}$$

for certain values of the real parameters $g_0, A, B,\ a_1, ..., a_m,\ b_1, b_2, ..., b_m, a_0,\ b_0$.

Clearly if this curve is non singular then the genus is $\mathcal{G} = \frac{1}{2}n(n-1)$.

Example 7.3

Let us suppose that the given algebraic curve (7.4) is such that

$$g(x,y) = g_0 + A\int(\prod_{j=1}^{m}x(x^2 - a_j^2))dx + B\int(\prod_{j=1}^{m}y(y^2 - b_j^2))dy.$$

It is possible to construct the non-Darboux integrable polynomial vector fields of degree $n = 2m + 1$. In particular for $n = 3, 5, 7$ we construct the following non-Darboux integrable polynomial systems.

For the polynomial system of degree seven

$$\begin{cases} \dot{x} =& y(160p^4y^2q^2 - 192p^2y^4q^2 + 64p^2y^6 - 96p^4y^4 + 32p^6y^2 - 64p^4q^4 - 32p^6q^2 - \\ & 96y^4q^4 - 32p^2q^6 + 32y^2q^6 + 160p^2y^2q^4 + 64y^6q^2)\nu_0 + \lambda y(-12p^2x^4 - 4p^4y^2 - 4y^2q^4 + \\ & 8p^2y^2x^2 + 4p^2y^4 + q^6 - p^2q^4 + 12x^2p^2q^2 + 8y^2x^2q^2 - 8p^2y^2q^2 + 8x^6 + 2x^2p^4 + \\ & 2x^2q^4 - 8y^4x^2 - 12x^4q^2 - p^4q^2 + p^6 + 4y^4q^2) \\ \dot{y} =& -1/64x(-64p^4q^4 - 32p^2q^6 - 96x^4p^4 + 32p^6x^2 + 32x^2q^6 - 32p^6q^2 - \\ & 192x^4p^2q^2 + 160p^4x^2q^2 + 160p^2x^2q^4 + 64p^2x^6 - 96x^4q^4 + 64x^6q^2)/(p^2+q^2)\nu_0 \\ & -1/64x\lambda(8p^2y^2x^2 - 12p^2y^4 - 8y^2x^4 - p^2q^4 - 8x^2p^2q^2 + 8y^2x^2q^2 + 12p^2y^2q^2 + 2p^4y^2 - \\ & 4x^2p^4 - 4x^2q^4 + q^6 + p^6 + 2y^2q^4 - p^4q^2 + 4p^2x^4 + 4x^4q^2 - 12y^4q^2 + 8y^6)/(p^2+q^2) \end{cases}$$

the invariant curve is

$$\begin{aligned} g(x,y) =& 1/8x^8 + (-1/4p^2 - 1/4q^2)x^6 + (1/8p^4 + 1/2p^2q^2 + 1/8q^4)x^4 - \frac{1}{4}(p^4q^2 + p^2q^4)x^2 + \\ & 1/8y^8 + (-1/4p^2 - 1/4q^2)y^6 + (1/8p^4 + 1/2p^2q^2 + 1/8q^4)y^4 + (-1/4p^4q^2 - \\ & 1/4p^2q^4)y^2 - 1/128q^8 + 1/32q^6p^2 + 13/64q^4p^4 + 1/32q^2p^6 - 1/128p^8 = 0 \end{aligned}$$

with the cofactor

$$K = \lambda(-y + x)(x + y)(-p^2 - q^2 + y^2 + x^2)yx.$$

In this example we have that

$$a_1 = -a_2 = p, \quad a_3 = -a_4 = q, \quad a_5 = -a_6 = \sqrt{\frac{1}{2}(p^2+q^2)}.$$

For the quintic vector field

$$\begin{cases} \dot{y} =& \nu_0(-3x^2 + r^2)(6r^2 - 6x^2)x + \lambda(-3x^2 + r^2)(-r^2 + y^2)y \\ \dot{x} =& -\nu_0(6r^2 - 6y^2)(r - 3y^2)y - \lambda(r^2 - x^2)x(r^2 - 3y^2) \end{cases}$$

The curve

$$g(x,y) = \frac{1}{6}y^6 - \frac{1}{3}r^2y^4 + \frac{1}{6}r^4y^2 + \frac{1}{6}x^6 - \frac{1}{3}x^4r^2 + \frac{1}{6}r^4x^2 - \frac{1}{39}r^6 = 0$$

is its invariant with the cofactor

$$K = 6y^2\lambda(3x^2 - r^2).$$

The cubic polynomial system

$$\begin{cases} \dot{x} =\nu_0(y^3-y)+\lambda x(2q^2x^2-2p^2y^2-3q^4+p^4) \\ \dot{y} =-\nu_0(x^3-x)+\lambda y(2q^2x^2-2p^2y^2+3q^4-p^4), \end{cases}$$

has as invariant curve of degree four

$$g(x,y)=1/8q^4+1/8p^4+1/4x^4-1/2q^2x^2+1/4y^4-1/2p^2y^2=0$$

with cofactor

$$K=4\lambda(2q^2x^2-2p^2y^2-q^4+p^4).$$

It is interesting to observe that the above constructed polynomial system of degree seven under the change

$$x=\sqrt{X},\quad y=\sqrt{Y},$$
$$\mathbf{V}=\Big(\frac{\mathbf{v}(x)}{y},\ \frac{\mathbf{v}(y)}{x}\Big)|_{x=\sqrt{X},y=\sqrt{Y}},$$

can be transformed to the cubic system

$$\begin{cases} \dot{Y} =(-64p^2-64q^2)X^3+(96q^4+192p^2q^2+96p^4)X^2+ \\ \quad (-32p^6-32q^6-160q^2p^4-160q^4p^2)X+64p^4q^4+32p^2q^6+32p^6q^2)p0+(-8Y^3+ \\ \quad (12p^2+12q^2)Y^2+(-2p^4-2q^4-8p^2X-12p^2q^2-8Xq^2+8X^2)Y+q^4p^2-p^6-q^6+ \\ \quad q^2p^4+8Xp^2q^2-4p^2X^2+4Xq^4+4Xp^4-4X^2q^2)\lambda) \\ \dot{X} =-((-64p^2-64q^2)Y^3+(96q^4+192p^2q^2+96p^4)Y^2+ \\ \quad (-32p^6-32q^6-160q^2p^4-160q^4p^2)Y+64p^4q^4+32p^2q^6+32p^6q^2)p0+8X^3+ \\ \quad (12p^2+12q^2)X^2+(-12p^2q^2-8p^2Y-8Yq^2+8Y^2-2q^4-2p^4)X- \\ \quad p^6-q^6+q^4p^2+q^2p^4-4Y^2q^2+4Yq^4+8p^2Yq^2+4p^4Y-4p^2Y^2)\lambda) \end{cases}$$

which admits the invariant curve of degree four

$$\begin{aligned} g(X,Y)=&1/8X^4+(-1/4p^2-1/4q^2)X^3+(1/2p^2q^2+1/8p^4+1/8q^4)X^2+ \\ &(-1/4q^2p^4-1/4q^4p^2)X+1/8Y^4+(-1/4p^2-1/4q^2)Y^3+(1/2p^2q^2+ \\ &1/8p^4+1/8q^4)Y^2+(-1/4q^2p^4-1/4q^4p^2)Y-1/128p^8+1/32p^6q^2- \\ &1/128q^8+13/64p^4q^4+1/32p^2q^6 \end{aligned}$$

with cofactor

$$K(X,Y)=(X-Y)(X+Y-(p^2+q^2))\lambda.$$

Example 7.4

Finally we analyze the case when

$$g(x,y)=g(x_0,y_0)+\int_{x_0}^{x}\prod_{j=1}^{m+1}x(x^2-a_j^2)dx+\int_{y_0}^{y}\prod_{j=1}^{m}y(y^2-b_j^2)dy$$

is a curve of degree $2m+4$.

Analogously to the above case we can construct a non-Darboux integrable polynomial vector fields of degree $n = 2m + 2$

In particular for $n = 4, 6$ we construct the following polynomial systems.

For $n = 4$

$$\begin{cases} \dot{x} = (1/6y^2x^2 - 5/108y^2 - 1/18x^4 + 7/324x^2 + 1/729)\nu_0 + (1/6xy^3 - 1/54xy)\lambda \\ \dot{y} = (-1/54xy + 1/4xy^3 - 1/12yx^3)\nu_0 + (1/4y^4 - 5/108x^4 - 1/18y^2 + 4/2187 + 4/243x^2)\lambda \end{cases}$$

the invariant curve is

$$g(x, y) = 1/6x^6 - 5/36x^4 + 2/81x^2 + 1/4y^4 - 1/18y^2 + 4/2187 = 0.$$

The cofactor in this case is

$$K = -\frac{1}{236196}(1458x^6 - 1215x^4 + 216x^2 + 2187y^4 - 486y^2 + 16)\lambda.$$

For $n = 6$ we construct the following vector field

$$\begin{cases} \dot{x} =(-1/128y^4x^2 - 4/3y^2 + 1/64x^6 - 1/2x^4 + 1/12y^4 + 40/9x^2 + 1/8y^2x^2 - 256/27)\nu_0 + \\ \quad (1/8xy^3 - 3/512xy^5 - 1/2xy)\lambda \\ \dot{y} =(-\frac{1}{96}xy^5 + \frac{5}{18}xy^3 + \frac{1}{48}yx^5 - \frac{4}{9}yx^3 + \frac{16}{27}xy)\nu_0 + \\ \quad (-\frac{1}{128}y^6 - \frac{11}{6}x^4 + \frac{1}{4}y^4 - \frac{1024}{27} - 2y^2 + 16x^2 + \frac{1}{16}x^6)\lambda \end{cases}$$

The invariant curve and its cofactor are respectively

$$g(x, y) = 1/6y^6 - 16/3y^4 + 128/3y^2 + 1/8x^8 - 16/3x^6 + 704/9x^4 - 4096/9\,x^2 + 65536/81 = 0,$$

$$K = (-4y + y^3 - 3/64y^5)\lambda + (1/8\,x^5 - 8/3\,x^3 - 1/16\,y^4\,x + y^2x + 32/3\,x)\nu_0.$$

We observe that after the change

$$x = X^2, \quad y = Y^2,$$
$$\mathbf{V} = \Big(\frac{\mathbf{v}(x)}{x}, \frac{\mathbf{v}(x)}{y}\Big)|_{x=X^2,\, y=Y^2}$$

the above system takes the form

$$\begin{cases} \dot{X} =Y((-1/2X^2Y^2 + 1/8X^2Y^6 - 3/512X^2Y^{10})\nu_0 + \\ \quad (-1/128Y^8X^4 + 1/64X^{12} - 256/27 - 4/3Y^4 + 40/9X^4 + 1/8Y^4X^4 - 1/2X^8 + 1/12Y^8)\lambda) \\ \dot{Y} =X((-1/128Y^12 - 2Y^4 + 1/4Y^8 + 1/16X^{12} - 11/6X^8 + 16X^4 - 1024/27)\nu_0 + \\ \quad (-1/96X^2Y^{10} - 4/9Y^2X^6 + 5/18X^2Y^6 + 1/48Y^2X^{10} + 16/27X^2Y^2)\lambda) \end{cases}$$

This polynomial system of degree 13 admit as invariant curve of degree 16

$$g(X,Y)=1/6Y^{12}-16/3Y^{8}+128/3Y^{4}+1/8X^{16}-16/3X^{12}+704/9X^{8}-4096/9X^{4}+65536/81=0$$

with cofactor

$$K=\frac{1}{96}(\nu_0(24X^{10}-512X^{6}+192Y^{4}X^{2}-12Y^{8}X^{2}+2048X^{2})+\lambda(192Y^{6}-9Y^{10}-768Y^{2})YX)$$

We observe that the constructed above curve are singular curves.

We state the following problem

Problema 7.2 To construct the differential system (7.3) for which the invariant curve (7.4) is non singular.

Clearly, if the curve (7.4) is non singular then the genus is

$$\mathcal{G}=\frac{1}{2}n(n-1).$$

as a consequence the maximum number of algebraic limit cycles are $\frac{1}{2}n(n-1)+1$.

To construct the require vector field should be satisfied the relation $dg(\mathbf{v})=K(x)g$ under the condition that the curve $g(x,y)=0$ is non singular. To obtain the explicit expression for the vector field in general it is necessary to solve a lot of technical and theoretical problems. In particular for the cubic system we have 31 parameters which must be satisfies 27 equations. By solving these equations we obtain that the nonsingular curve of genus 3:

$$\begin{aligned}g(x,y)=&1/4x^4-1/3(a_2+a_1)x^3+1/2a_2a_1x^2+1/4y^4-1/3(b_2+b_1)y^3+1/2b_2b_1y^2+\\&1/12\mu\Big(-5b_2^3b_1^3a_2^2-5b_2^3b_1^3a_1^2+2b_2^4b_1^2a_1^2+2b_2^2b_1^4a_1^2-2b_1a_1^2a_2^4b_2+2b_2^4b_1^2a_2^2+\\&2b_2^2b_1^4a_2^2-2b_2^2b_1^4a_1a_2+2a_1^4a_2^2b_2^2+5b_2^3b_1^3a_1a_2-2b_2^4b_1^2a_1a_2+\\&5b_1a_1^3a_2^3b_2+2b_1^2a_1^2a_2^4-5a_1^3a_2^3b_2^2-2b_1a_1^4a_2^2b_2+2a_1^2a_2^4b_2^2-5b_1^2a_1^3a_2^3+2b_1^2a_1^4a_2^2\Big)=0,\\&\mu\equiv\Big((a_1^2-a_1a_2+a_2^2)(-b_2b_1+b_2^2+b_1^2)\Big)^{-1}\end{aligned}$$

which is invariant of the cubic system if the parameters a_1, a_2, b_1, b_2 are roots of the certain homogenous polynomials $P_j(a_1,a_2,b_1,b_2),\ j=1,2,3$ such that

$$\begin{aligned}&P_j(\lambda a_1,\lambda a_2,\lambda b_1,\lambda b_2)=\lambda^8P_j(a_1,a_2,b_1,b_2),\quad j=1,2\\&P_3(\lambda a_1,\lambda a_2,\lambda b_1,\lambda b_2)=\lambda^{24}P_3(a_1,a_2,b_1,b_2)\end{aligned}$$

8. The Poincaré problem and 16th Hilbert problem for algebraic limit cycles

8.1 The Poincaré problem.

The question on the existence an effective procedure to find a natural number $\mathcal{N}(n)$ which bounds the degree of all irreducible invariant curve of a non-Darboux integrable polynomial system of a degree n is well known as Poincaré problem [Poin, Car, Sad1].

The problem on the existence of the polynomial planar systems with an invariant algebraic curve of the maximum degree was studied in particular in [Chr, Sad2]. In [Chr] the author gave an explicit polynomial system of degree n for each non-singular real algebraic curve $g = 0$ of degree n what is that system's invariant. The author states also that n is optimal for a generic class of algebraic curve.

Proposition 8.1

Let $\mathcal{N}(n)$ is the maximum degree of the irreducible algebraic curve (4.10) that is invariant curve of the polynomial system (3.5) of degree n.

Then

$$(n+1) \leq \mathcal{N}(n) \leq 2n(n+1)$$

The proof of the lower bound follows from the proposition 7.2 and 7.3 and the upper bound from the proposition 5.4. The upper bound is reached in particular for $n = 2$.

8.2 The 16th Hilbert Problem for Algebraic Limit Cycles

In 1900 Hilbert [Hil] proposed in the second part of his 16th problem to estimate a uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree. This question has been studied in particular in [Sad1, Sad2] for algebraic limit cycles.

By considering that the ovals of the invariant algebraic curve are isolated periodical solutions of the vector field for which is it invariant we deduce that the maximum number of algebraic limit cycles of the polynomial system of degree n with one invariant curve (we denote this number by $A(n,1)$) is at most the *genus of the curve +1,* i.e.,

$$A(n,1) \leq \mathcal{G} + 1$$

hence we observe that to solve the 16th Hilbert's Problem for Algebraic Limit Cycles it is necessary firstly to solve the Poincaré problem for this case, i.e.,it is necessary to find the maximum degree of the invariant curve [Sad1].

Proposition 8.2

$$A(n,1) \leq (2n^2 + 2n - 1)(n^2 + n - 1) + 1.$$

The proof follows from the proposition (5.4) and from the Harnak theorem on the maximum numbers of the ovals of the algebraic curve.

From the above results we prove the following

Proposition 8.3

The maximum number of the algebraic limit cycles for the polynomial planar vector field of degree n with one invariant algebraic curve $A(n,1)$ is such that

$$A(n,1) \geq max\big(h_0(n), h_1(n), h_2(n), h_3(n), h_4(n),\big)$$

where $h_j(n)$ are the maximum numbers of ovals of the algebraic curves $H_j(x,y) = 0$:

$$
\begin{aligned}
&H_0(x,y) = \sum_{m=0}^{n} a_{n-m\,m} x^m y^{n-m},\\
&H_1(x,y) = a x^{n+1} + \sum_{m=0}^{n-1} a_{n-m\,m} x^m y^{n-m},\\
&H_k(x,y) = \int (\prod_{1}^{m_k} (x-a_j)dx + \prod_{1}^{l_k} (y-b_j)dy),\\
&a_1 < a_2 .... < a_{m_k}, \quad b_1 < b_2 .... < b_{l_k}, \quad k = 2,3,4\\
&m_2 = l_2 = n-1, \quad m_3 = n+1, \quad l_3 = n-1, \quad m_4 = n,\ l_4 = n.
\end{aligned}
$$

From this inequality we deduce the following result[Sad2];
Corollary 8.1

$$A(n,1) \geq \frac{1}{2}(n-1)(n-2) + 1.$$

Conjecture

$$A(n,1) \geq \frac{n(n-1)}{2}$$

This conjecture can be solve if we construct the polynomial system of degree n with one invariant nonsingular irreducible invariant curve of degree $n+1$ (see proposition 8.1).

It is interesting to observe that in [Llib1] stated the following problems:

Is 1 the maximum number of algebraic limit cycles that a quadratic system can have?.

Is 2 the maximum number of algebraic limit cycles that a cubic system can have?.

The answer to this questions for the system (3.5) with $S > 1$ was given in [Sad2].

Is there a uniform bound for the number of algebraic limit cycles that a polynomial vector field of degree n could have?.

What is the maximum degree of an algebraic limit cycle of a quadratic polynomial vector field?.

What is the maximum degree of an algebraic limit cycle of a cubic polynomial vector field?

The partial solutions of these problems we obtain from the above results and results proposed in [Sad1, Sad2].

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