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Editorial

Nonlinear Dynamics on Interconnected Networks

1. Introduction

Networks of dynamical interacting units can represent many complex systems, from the human brain to transportation systems and societies. The study of these complex networks, when accounting for different types of interactions has become a subject of interest in the last few years, especially because its representational power in the description of users' interactions in diverse online social platforms (Facebook, Twitter, Instagram, etc.) [1], or in representing different transportation modes in urban networks [2,3]. The general name coined for these networks is multilayer networks, where each layer accounts for a type of interaction (see Fig. 1).

In the last years, it has been a boosted interest in the analysis of the structure and dynamics on multilayer networks [4–7], essentially because the well-known theory of complex networks developed for the study of a single layer network has to be revisited when analyzing multilayer networks. Moreover, the outcome of the analysis has revealed that new emergent physical phenomena can appear as a direct consequence of the multilayer structure. In particular, the analysis of the robustness of the structure in the presence of perturbations or defects, as well as the cascade propagation of failures has focussed the analysis of important contributions in the field [4,8–13].

From the physics point of view, the study of simple diffusion processes (or more complex like epidemic spreading, etc.) has driven the understanding of the interplay between dynamical processes and structure in the subject [11,14–16]

From the mathematical point of view, the new level of complexity required the definition of a novel mathematical framework [17], based on tensorial algebra, for their representation and their structural reduction to simpler subsets of networks [18]. The outgrowth of these mathematical approaches is the development of new structural descriptors, from centrality measures [19–24] to partitions in communities to describe the mesoscale organization of nodes in multilayer topologies [25–27].

2. Overview of papers in the special issue

In the following, we will briefly review the studies accompanying this Special Issue. In a few cases, for sake of completeness, we report some mathematical details that have been casted to a common notation, to avoid confusion to the reader. More specifically, we will use Latin lower indices to indicate nodes and Greek upper indices to indicate layers.

Uncovering correlations in networked systems is crucial to understand empirical networks. By exploiting the tensorial formulation of multilayer networks [17], Ferraz de Arruda, Cozzo, Moreno and Rodrigues [29] proposed a generalization of the concept of assortativity that can be used in directed and weighted networks. Projecting the multilayer adjacency tensor $M^{i\alpha}_{j\beta}$ – accounting for both intra- and inter-layer connections among node i in layer α and node j in layer β – into different subspaces, they were also able to unveil correlations of each layer separately and of the underlying network of layers. They applied this new methodology to the network of European airports, where each layer represents an airline, and found that aggregated representation of this multilayer network might exhibit very different correlation patterns that might lead to an incorrect understanding of the system. Their second application concerned the effects of correlations on epidemics spreading, where they introduced the tensor

$$R^{i\alpha}_{j\beta}(\lambda,\gamma) = M^{i\eta}_{j\sigma}E^{\sigma}_{\eta}(\alpha\beta)\delta^{\alpha}_{\beta} + \frac{\gamma}{\lambda}M^{i\eta}_{j\sigma}E^{\sigma}_{\eta}(\alpha\beta)(U^{\alpha}_{\beta} - \delta^{\alpha}_{\beta}), \quad (1)$$

generalizing the well-known contact matrix [30] to the case of interactions between node i in layer α and node j in layer β, where E, U and δ are special tensors, λ and γ are epidemics parameters encoding the probability of spreading through an intra-layer contact and the spreading probability through an inter-layer contact, respectively. Their microscopic Markov chain model for the diffusion of the epidemics suggest that correlations have a larger impact on the spreading dynamics when the coupled networks have similar levels of heterogeneity and that, at variance with disassortative networks, assortative multilayer systems exhibit a smaller epidemic threshold, with the disease showing a faster initial growth rate but a shorter duration.

Another dynamics of interest for applications, the one of opinions, can be modeled within a multilayer framework where each layer represents a different topic. Battiston, Cairoli, Nicosia, Baule and Latora [31] proposed a model where agents can have different opinions on different topics and, additionally, they can be subjected to media pressure. By imaging the opinion of one agent as an arrow that can change its orientation, agents can be coherent or incoherent, depending on the fact that they have the same orientation on different layers or not, respectively. The ith agent in layer α is modeled as a particle with spin s^{α}_i tending to spread its own ideas to agent j in the neighborhood while, simultaneously, it is affected by mass media, here playing the role of external fields h^{α} applied to the spin system in each layer. The functional governing the dynamics of each agent i in layer α is given by

$$F^{\alpha}_i = J\sum_{j=1}^{N} a^{\alpha}_{ij}s^{\alpha}_j + \gamma\frac{\chi_i}{J}\sum_{\beta=1}^{M} s^{\beta}_i\delta^{\alpha}_{\beta} + h^{\alpha}, \quad (2)$$

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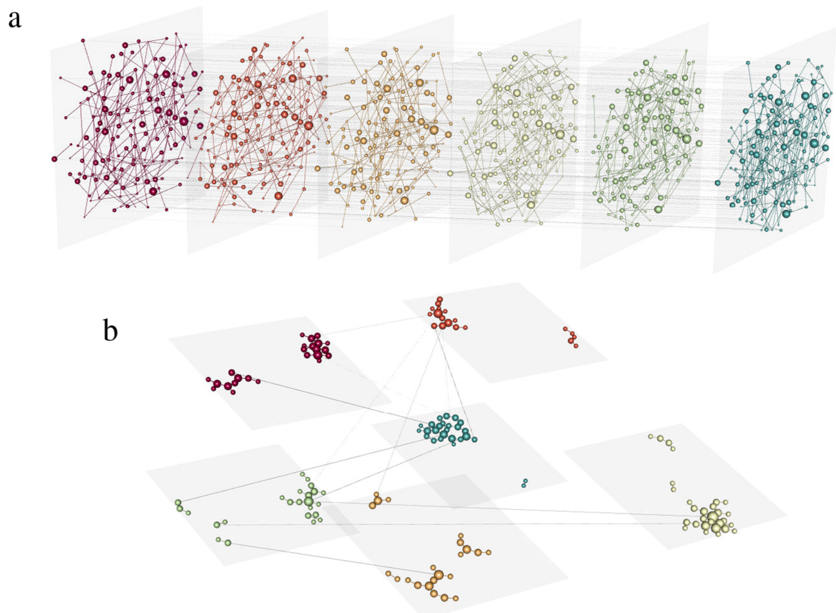

Fig. 1. Illustration of multilayer networks of different types. In both cases, the number of nodes is 180 and the number of layers is 6, but the topology is dramatically different: (a) a multiplex networks where a node exists in one or more layers and inter-layer links connect its replicas; (b) an interdependent network, also known as network of networks, where each node exists only in one layer and inter-layer links connect different nodes on different layers. This visualization has been created with muxViz [28].

where a_{ij}^{α} encodes the connections between agents in layer α, J is a coefficient which models the intrinsic permeability of agent i to social pressure and δ_{β}^{α} is the Kronecker delta function. On the right-hand side of Eq. (2), the first term models the social pressure exerted on i by its neighbors, whereas the second term models the tendency of agent i towards internal coherence. The parameter γ tunes the relative importance of internal coherence and social pressure, while χ_i determines the importance of internal agent coherence. A rich variety of consensus patterns can be modeled by using this approach. By introducing thermal noise, it has been shown that global consensus can be achieved only below a critical temperature and that mass media can be used to polarize the consensus of the whole system on a specific opinion.

A random rectangular graph (RRG) is a generalization of the random geometric graph (RGG) in which the nodes are embedded into a rectangle with side lengths a and $b = 1/a$, instead of on a unit square $[0, 1]^2$. Two nodes are then connected if and only if they are separated at a Euclidean distance smaller than or equal to a certain threshold radius r. This particular network structure is usually intended to represent urban street networks where the nodes describe the intersection of streets, represented by the edges of the graph. These streets and their intersections are embedded in the two-dimensional space representing the surface occupied by the corresponding city. Similar situations occur with infrastructural and transportation systems ranging from water supply networks and railroads to the internet and wireless sensor networks. Estrada has found [32] a lower bound for the diameter of RRGs. The diameter is an important parameter *per se* as well as for its inclusion in many inequalities for other network's structural parameters. Moreover, he uses this bound to find an upper bound for the algebraic connectivity of RRGs. The algebraic connectivity, the second smallest eigenvalue of the graph Laplacian, is one of the most important parameters relating network structure and dynamical processes taking place on them, e.g., consensus/diffusion dynamics, synchronization. Finally, the contribution focuses on the consensus dynamics on RRGs where he finds analytically that as the rectangle becomes more elongated, the time for reaching consensus increases polynomially with the side length of the rectangle.

Why and how cooperative interactions thrive at all levels of organization, from human societies to the simplest biological systems, has obtained great research attention across many disciplines. Complex network theory has provided important insights into the dynamics of interactions in a structured population, and the updating rules are the key process for the evolution of strategy behaviors in the framework of evolutionary dilemmas. Until now, the discussion on evolutionary games on networks has mainly focused on network structure and the nature of the game. Zhang and Chen [33] scrutinize the role of strategy updating by proposing a new way to confront the analysis. The idea is to put the attention to the players' switching probabilities. In the theoretical contact-based setup proposed, players need to know their neighbors' exact payoff information, detouring the requirement for explicit information of the related payoffs, irrespective of the strategy types. Employing the players' switching probabilities as a key step to establish the evolution of strategies in the long run they compute the outcome of the the classical game theoretical dilemmas.

Coupled infrastructures such as power grid, Internet, water and gas distribution, etc. are widely investigated because their robustness to overload and their resilience to targeted or random failures might affect the lives of millions of people. Scala, De Sanctis Lucentini, Caldarelli and D'Agostino [34] investigated the abrupt breakdown behavior of coupled distribution grids under load growth. They considered the case of several coupled networks and study the cascading behavior of such a model under increasing stress, possibly driven by other layers. An emblematic example is given by the increase in the cost of gas experienced by Ukraine that increased the stress on the electrical network, a cheaper energy vector. In their model, when a system α is subjected to some failures it decreases its load by increasing the load on all other layers $\beta \neq \alpha$ by $l^{\alpha} f^{\alpha} T_{\alpha\to\beta}$, where f^{α} and l^{α} are the fraction of failed links and the load per link, respectively, in system α. Their mean-field approach resulted in a system of coupled equations

$$f^{\alpha}(t+1) = P^{\alpha}(\tilde{l}^{\alpha}(t)/(1 - f^{\alpha}(t))), \quad (3)$$

being $\tilde{l}^{\alpha}(t)$ the load per link experienced by layer α at the t-th stage of the cascade and $P^{\alpha}(x)$ is the cumulative distribution function accounting for the capacity of system α. They found evidence for first-order transitions and their findings suggest that two competing effects emerge while increasing the coupling among the systems: the safety region where grids can operate without withstanding systemic failures is enlarged, although when systems fail they tend to fail together.

When trying to predict the long-term behavior in networks of interacting units, a recurrent question is to characterize collective properties, such as synchronization and predictability, in terms of the network topology and interaction strengths. Skardal, Taylor, Sun and Arenas [35] investigated the dynamics of network-coupled phase oscillators in the presence of heterogeneous coupling frustration, where the interactions between different pairs of network neighbors are allowed to be described by different functions. They predicted, analytically, the behavior of some well-known networked system, and showed that at variance with homogeneous coupling, the presence of heterogeneity amplifies the total erosion of synchronization while increasing the deviation from the perfectly synchronized state. Their findings provided evidence that the synchronized solution remains stable for smaller ranges of coupling frustrations when heterogeneity is allowed.

Fernandez and Blumenthal [36] focused their study on networks of coupled degrade-and-fire (DF) oscillators, which are simple dynamical models of assemblies of interacting self-repressing genes. Through a deep mathematical analysis, they have found that periodic and exhaustive cycle of firings implies asymptotic periodic behavior of gene expression levels, i.e. trajectory behaviors are uniquely determined by their firing cycle. The implications of such finding are sound because, independently on the topology, the asymptotic periodic behavior entrained by a firing sequence is the fate of systems of interacting DF oscillators. The result still has to be proved, but the conjecture is there.

The study of bipartite networks is of utmost importance to unravel characteristics of networks formed by nodes of different two-types. This is very common in networked systems where the representation of different entities is needed, for example users and usage (e.g. dvd renters and dvd movies, etc.). Estrada and Gomez-Gardeñes have proposed a mathematical framework to attack the complexity behind their analysis [37]. They defined a network spectral bipartivity index that proved to be very useful in the analysis of such relations. The proposed measure is tested by analyzing the European air transportation system, represented by 33 passenger airlines, 25 of which correspond to “traditional” and 8 to “low-cost” carriers (LCCs). For each airline carrier they consider a network in which the nodes representing any of the 450 commercial airports existing in Europe and two nodes (airports) are connected if the corresponding airline has a flight between them. By using the spectral bipartivity index, their results show the organizational differences of these two types of airlines (traditional and low-cost) and how the alliances between traditional airlines affect the value of bipartivity. Motivated by the above findings, they observe that traffic efficiency (as revealed from real data about the traffic flow of each airline) is strongly and negatively correlated with the bipartivity of its network. Therefore, bipartivity seems to provide a good descriptor of the efficiency of transportation networks and can be used to test the goodness of alliances and possible mergers of airlines.

Cascading failures in complex networks have been the subject of intensive research, because of their importance in practical applications, from power grids overloading to traffic congestion in road networks. In general, the dynamics of the cascade is governed by several factors, among which the interactions between different layers. Burkholz, Leduc, Garas and Schweitzer [38] presented a study about cascade failures on a two-layer multiplex network where the feedback depends on the coupling strength between the layers. Their results are relevant for coupled networks like interbank systems, where banks are exposed to each other via different types of business activities. In this framework, the authors developed a model where failures play the role of bankruptcies: the first layer encodes firms’ exposure in the core business whereas the second layer represents exposures between firms in the subsidiary business. The cascading dynamics on each layer is according to the threshold failure mechanism proposed by Watts [39], where a node i fails in layer α if a fraction of its neighbors failed and its fragility is above a predefined threshold θ_i^α. The fragility of a node i in layer α is given by

$$\phi_i^\alpha = \frac{n_i^\alpha}{k_i^\alpha}, \tag{4}$$

where n_i^α is the number of its failed neighbors and k_i^α is its degree, in layer α. When the coupling strength between the two layers is small, they have shown that the systemic risk in the multiplex network is smaller than in the aggregated one. They observed sharpened phase transitions in the cascade size that are less pronounced on the aggregated representation of the system, with systemic risk increasing above a critical coupling strength because of the mutual amplification of cascades in the two layers.

In many diffusion processes, such as epidemics spreading in spatial networks, information dissemination in social networks or traffic in transportation networks, it is of crucial importance to identify nodes that are more central than others in the system, with respect to certain criteria or specific dynamics. Solé-Ribalta, De Domenico, Gómez and Arenas [40] propose a method to identify such nodes in multilayer and interconnected systems based on random walk diffusion through the network. By exploiting the tensorial algebra [17] to represent multilayer networks and random walk dynamics on the top of such systems [11] they first calculate quantities of interests such as node’s occupation probability and mean first passage time between any pairs of nodes, providing analytical expressions. Their calculations make use of the concept of absorbing states, where the destination node of a walker is assumed to trap the walker forever. By introducing the transition tensor T governing random walks over multilayer networks and the corresponding absorbing transition tensor $T_{[d]}$, their calculations are based on the tensor

$$p_{j\beta}^{o\sigma}(t) = \left(T_{[d]}^t\right)_{j\beta}^{o\sigma} \tag{5}$$

indicating the probability of visiting node j in layer β, after t time steps, considering that the walk originated in node o in layer σ. This mathematical trick allows them to calculate efficiently random walk betweenness and closeness centralities. Random walk closeness centrality of a node i in a multilayer network is defined as the inverse of the average number of steps that a random walker, starting from any other node in the multilayer network, requires to reach any replica of node i for the first time. Random walk betweenness of a node i is defined as the amount of random walks between any pair of origin and destination nodes that pass through any replica of node i in the multilayer network. Their findings, completing this Special Issue, are useful to identify central nodes in real systems where entities that travel the network do not always take the shortest path.

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