Enhancement of signal response in complex networks induced by topology and noise

Juan A. Acebrón, Sergi Lozano, and Alex Arenas

Abstract The effect of the topological structure of a coupled dynamical system in presence of noise on the signal response is investigated. In particular, we consider the response of a noisy overdamped bistable dynamical system driven by a periodic force, and linearly coupled through a complex network of interactions. We find that the interplay among the heterogeneity of the network and the noise plays a crucial role in the signal response of the dynamical system. This has been validated by extensive numerical simulations conducted in a variety of networks. Furthermore, we propose analytically tractable models based on simple topologies, which explain the observed behavior.

1 Introduction

The *stochastic resonance effect*, first conceived as a plausible mathematical explanation of the phenomenon of *glacial cycles*, has deserved an important part of the applied mathematics and physics literature of the last thirty years. It represents the interesting effect manifested in subthreshold nonlinear dynamics where a weak input signal can be amplified by the assistance of noise. Such a favorable result can be quantitatively explained by the matching of two time scales: the period of the

Juan A. Acebrón
Departament d'Enginyeria Informatica i Matematiques, Universitat Rovira i Virgili, e-mail: juan.acebron@urv.cat

Sergi Lozano
Departament d'Enginyeria Informatica i Matematiques, Universitat Rovira i Virgili, e-mail: sergio.lozano@urv.cat

Alex Arenas
Departament d'Enginyeria Informatica i Matematiques, Universitat Rovira i Virgili, alexandre.arenas@urv.cat

input signal (the deterministic time scale) and the Kramers rate (i.e. the inverse of the average switch rate induced by the sole noise: the stochastic time scale) [1].

General statements about the necessary conditions for the emergence of stochastic resonance are usually related to three main aspects: (i) a dynamic non-linear system endowing a potential with energetic activation barriers, usually bistable systems (ii) a small amplitude (usually periodic) external signal, and (iii) a source of noise inherent or coupled to the system. Neverthless, the response of the system can be enhanced by coupling it through several configurations, all-to-all, next neighbors [3], etc. This has been done mainly resorting to a linear coupling among oscillators.

In [9] we showed that this phenomenon is not restricted to the presence of stochastic fluctuations in the system, giving evidence that an equivalent amplification of a external signal can be obtained in a deterministic system of coupled bistable potentials in complex heterogeneous networks, producing what we termed topological resonance. Here, our goal is to show that the presence of noise can improve further the response of the system, by combining the topological resonance with the classical stochastic resonance phenomena.

The paper is organized as follows: in Sec. 2, we present the system model to be analyzed. The noiseless case is reviewed in Sec. 3. In Sec. 4, the role played by synchronization in the phenomenon is carefully analyzed. This is followed in Sec. 5 by an investigation of the effect of noise on the coupled dynamics. Finally, our results are summarized and discussed in Sec. 6.

2 Model equations

The prototype system, where the stochastic resonance phenomenon has been carefully analyzed, is represented by the overdamped motion of a particle in a bistable potential subject to a periodic signal in presence of random fluctuations. In its simple form, it consists in the following dimensionless equation

$$\dot{x} = x - x^3 + A\sin(\omega t) + \eta(t) \tag{1}$$

where the bistable potential is $V(x) = -x^2/2 + x^4/4$, the external signal $A\sin(\omega t)$ and $\eta(t)$ a Gaussian white noise with $< \eta(t) >= 0$, and $< \eta(t)\eta(t') >= 2D\delta(t - t')$.

The deterministic system we will study corresponds to a network of elements obeying Eq.(1). The network is expressed by its adjacency matrix A_{ij} with entries 1 if i is connected to j, and 0 otherwise. For simplicity, from now on we will consider only undirected unweighted networks. Mathematically the system reads

$$\dot{x}_i = x_i - x_i^3 + A\sin(\omega t) + \lambda L_{ij} x_j + \eta_i(t) \qquad i = 1, ..., N, \tag{2}$$

where $L_{ij} = k_i \delta_{ij} - A_{ij}$ is the Laplacian matrix of the network, being $k_i = \sum_j A_{ij}$ the degree of node i. Here, $\eta_i(t)$ are again Gaussian white noises uncorrelated among them.

3 Model without noise

In this section, we review the role played by the heterogeneity in the dynamical behavior of the system, already presented in [9]. To this purpose, and to avoid mixing of nonlinear effects, the noise term has been neglected.

3.1 Numerical results

Numerical experiments was performed over two classes of networks, considered representative of homogeneous and heterogeneous networks. For the former, an all-to-all connectivity is chosen, while for the latter the Barabasi-Albert (BA) network. Such a network model is probably the most studied growing model, which provides a scale-free (power-law) degree distribution for the degree of its nodes [5].
The mean gain G over different initial conditions was computed as function of the coupling λ, being the gain defined as $G = \max_i x_i/A$. The results obtained in [9] showed that the scale-free networks present a clear amplification of the external signal, for a significant range of values of λ that depends on the average degree, however the all-to-all connectivity does not amplify the signal.

3.2 Analytical treatment

To understand the observed behavior, we analyzed a simple topology consisting on a star network. Such a topology is simple enough to be mathematically tractable, and simultaneously capable to capture the heterogeneity found in scale free-type networks. In fact, the resulting dynamical system can be seen as composed of two parts: the dynamic of the hub (the highly connected node in the network with $N-1$ links), x_H, and the remaining nodes, y_i, linked to the hub. Then, the dynamical system (2) can be written out as

$$\dot{x}_H = [1-\lambda(N-1)]x_H - x_H^3 + A\sin(\omega t) + \lambda \sum_{i=1}^{N-1} y_i, \tag{3}$$

$$\dot{y}_i = (1-\lambda)y_i - y_i^3 + A\sin(\omega t) + \lambda x_H, \qquad i = 1,\cdots,N-1. \tag{4}$$

If the coupling λ is sufficiently small, the dynamic of the nodes can be decoupled from that of the hub, obtaining the well known equation of an overdamped bistable oscillator. In absence of forcing, the system possesses two stable fixed points centered around ± 1, as a first approximation, which corresponds to the minimum of the potential energy function $V_i(x) = -y_i^2/2 + y_i^4/4$. When the amplitude of the forced signal is subthreshold, each node i oscillates around the minimum of its potential with the same frequency of the forcing signal. Then, Eq. (3) can be solved for the

ith node, and asymptotically for long time yields,

$$y_i(t) \sim \xi_i - \frac{A}{\omega^2+4}\left[\omega\cos(\omega t) - 2\sin(\omega t)\right], \quad t \to \infty \tag{5}$$

with $\xi_i = \pm 1$ depending on the initial conditions. Inserting the solution above into the equation governing the dynamic of the hub, it can be rewritten as

$$\dot{x}_H = -V_H'(x_H) + A'\sin(\omega t) + B'\cos(\omega t) + \lambda\eta_i, \tag{6}$$

where $\eta_i = \sum_{i=1}^{N-1}\xi_i$, and $A' = A[1+2\lambda(N-1)]/(\omega^2+4)$, $B' = -2\lambda(N-1)/(\omega^2+4)$. Here $V_H(x,\eta) = -[1-\lambda(N-1)]x^2/2 + x^4/4 - \lambda x\eta$ represents the effective potential felt by the hub. Notice that the problem has been reduced to the motion of an overdamped oscillator in an effective potential driven by a reamplified forcing signal coming from the global sum of the remaining nodes.

From the equation above, two important facts can be observed. Firstly, the clear separation of scales governing both subsystems. In fact, a slight changes of coupling, decreases dramatically the potential barrier height for the hub, $h = [1-(N-1)\lambda]^2/4$, while remains almost constant for the remaining nodes. Moreover, when $\lambda = 1/(N-1)$ the barrier for the hub disappears, leading to a unique single fixed point $X_H = 0$. Secondly, the two possible solutions for the nodes, namely oscillations around ± 1, affect the dynamic of the hub as it were induced by a quenched disorder [6]. Indeed, in the limit $N \to \infty$, and by the central limit theorem, η_i behaves as a random variable governed by a gaussian probability distribution with variance $\sigma^2 = N$. This is because the initial conditions were randomly chosen. Note that both mechanisms may cooperate to allow the hub to surmount the potential barrier, and both are due mainly to the heterogeneity present in the network.

When $\lambda \neq 1/(N-1)$, Eq. (6) can be solved in the asymptotic long-time limit, carrying out the same calculation as before for the remaining nodes. This can be done in absence of forcing considering that now the hub relaxes to the equilibrium points, $x_H^{(0)}$. Such points are the minimum of the effective potential, and can be determined solving the cubic equation $V_H'(x_H^{(0)}) = -[1-\lambda(N-1)]x + x^3 - \lambda\eta$, being $V_H''(x_H^{(0)}) = -[1-\lambda(N-1)] + 3(x_H^{(0)})^2 > 0$. Therefore, as a first approximation, the resulting dynamics for the hub corresponds to oscillations around the equilibrium point with lowest potential value, being described by

$$\dot{\bar{x}}_H = -V_H''(x_H^{(0)})\bar{x}_H + A'\sin(\omega t) + B'\cos(\omega t), \tag{7}$$

where $\bar{x}_H = x_H - x_H(0)$. Hence, the solution is easily found, and yields

$$x_H(t,\eta) \sim x_H^{(0)} - \frac{1}{\omega^2 + a_H^2} \tag{8}$$
$$\{[B'\omega - A'a_H]\sin(\omega t) - [A'\omega + B'a_H]\cos(\omega t)\}, \quad t \to \infty$$

where $a_H = V_H''(x_H^{(0)})$. Note that $x_H(t,\eta)$ depends implicitly on the random variable η through the equilibrium point $x_H^{(0)}$. Knowing the long time evolution for the hub, its gain can be readily evaluated, and the result is

$$G(\eta) = \frac{1}{A}\frac{1}{a_H^2+\omega^2}\sqrt{(B'\omega - A'a_H)^2 + (A'\omega + B'a_H)^2} \tag{9}$$

In practice, to cancel out the dependence on the initial conditions, averaging over them is required. However, this turns out to be equivalent to averaging over η_i. Therefore, the mean gain can be estimated, and is given by

$$< G >= \frac{1}{\sqrt{2\pi N}}\int_{-\infty}^{\infty} d\eta\, e^{-\frac{\eta^2}{2N}} G(\eta). \tag{10}$$

The influence on the size of the network, N can be analyzed resorting to the equation above. In particular, we are interested in the case of N large, and for $\lambda = 1/(N-1)$, when the mean gain attains its maximum value. Making use of the Laplace method, it is possible to find an asymptotic solution to Eq. (10), in the limit of infinitely many nodes, $N \to \infty$, which is

$$< G >= G(0) + \frac{1}{N}\left.\frac{d^2G}{d\eta^2}\right|_{\eta=0} \eta + O(N^{-2}), \tag{11}$$

Therefore, the maximum mean gain can be extracted from a network, turns out to be bounded from above by $G(0)$. This simple analysis was shown to match pretty well the results of the numerical experiments in [9].

3.3 Extension to scale-free networks

Although a scale-free network is a much more intricate structure than the star network, it shares the finger print of heterogeneity namely the existence of hubs. A neat picture can be given considering that each highly connected node acts locally as it were the hub of a star network, with a degree k picked up from the degree distribution. Therefore, for a given coupling λ, we can find several star-like networks in different stages, depending on the degree of its local hubs. Recall that when $\lambda = 1/k$, the maximum signal amplification is attained for a hub with degree k. Furthermore, increasingly larger values of λ would activate local hubs with smaller degrees. In other words, the corresponding hub would provide the maximum gain of the network for such a coupling. Fig. 1 shows such a predicted behavior in case of a BA network with average degree $< k >= 3$. Taking into account that the network possesses several hubs, it should exist a wide range values of λ for which the maximum gain is achieved. This contrasts with the results found for the star network, and anticipates what we already pointed out for scale-free networks in [9], showing that

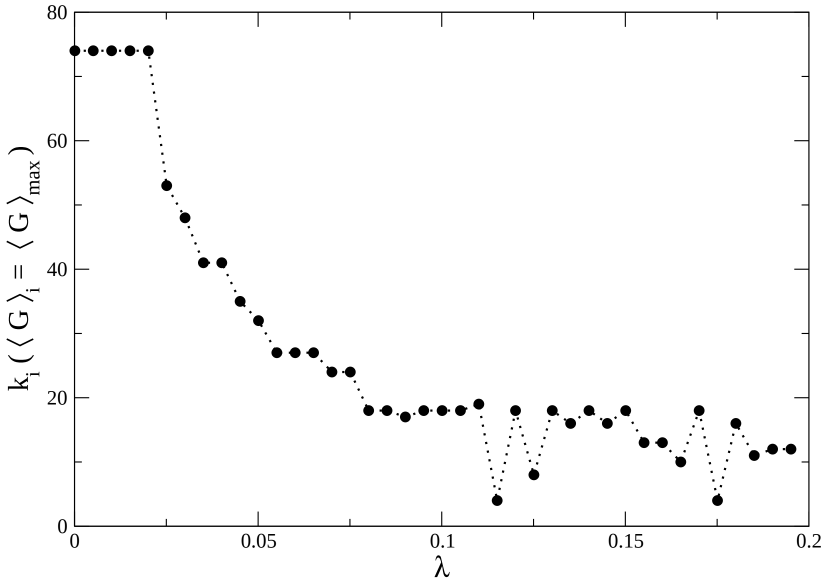

Fig. 1 Degree of the node in a BA network which provides the maximum mean gain $< G >$, as function of the coupling. Here $N = 500$,$A = 0.01$, and $\omega = 2\pi \times 10^{-1}$.

such a network consists on a much more robust topology where the phenomena can be found.

4 The role of synchronization

As opposed to other nonlinear coupled effects reported in literature [3], where synchronization seems to play a beneficial role in signal amplification, in our case it is responsible for deteriorating it. In [9], we showed that the mechanism described above remains valid until a full synchronization occurs. In the limit of large coupling, the nodes become tightly connected, synchronize and behave as a single node. By synchronization, here we mean spatial synchronization, measured it as the number of nodes occupying the two possible equilibrium points (the positive, n_+ and the negative one, n_-). Such a degree of synchronization, conveniently normalized by the number of nodes, was depicted in [9]. In particular, full synchronization was attained once a critical coupling was surpassed. Moreover, numerical simulations revealed that the path to synchronization is more pronounced for scale-free networks with lower average degree. Then, for this case a smaller range of values for which maximum gain is sustained was observed. Notably similar curves have been found in the prototype Kuramoto model [7], which describes synchronization phenomena in nonlinear coupled oscillators, on top of complex networks [8].

5 Model with noise

Having found that the heterogeneity of the network can induce a signal amplification, a natural concern arises on the robustness of the phenomena when random fluctuations are taken into account. To this purpose, we performed some numerical experiments on the model equation (2), computing the average amplification $< G >$, as well as the the single-to-noise ratio (SNR) for several values of the noise and coupling. This has been done mainly for the star-like network configuration. We integrate numerically the stochastic differential equation (2) using the Euler-Maruyama scheme, and compute the power spectral density for the hub, which we know that provides the maximum signal amplification. The power spectrum was obtained averaging 128 noise realizations of 2^{20} time steps each. The SNR is defined here as the ratio of the signal power divided by the noise power in the signal bin. The signal power has been computed by substracting the noise background from the total power in the signal bin. The chosen amplitude for the input signal now is larger than in Fig. 1, to facilitate the interwell potential motion. In this way, the amount of noise neccesary to surmount the barrier potential is reasonably small, and thus the computational cost required to reduce the statistical error is reduced.

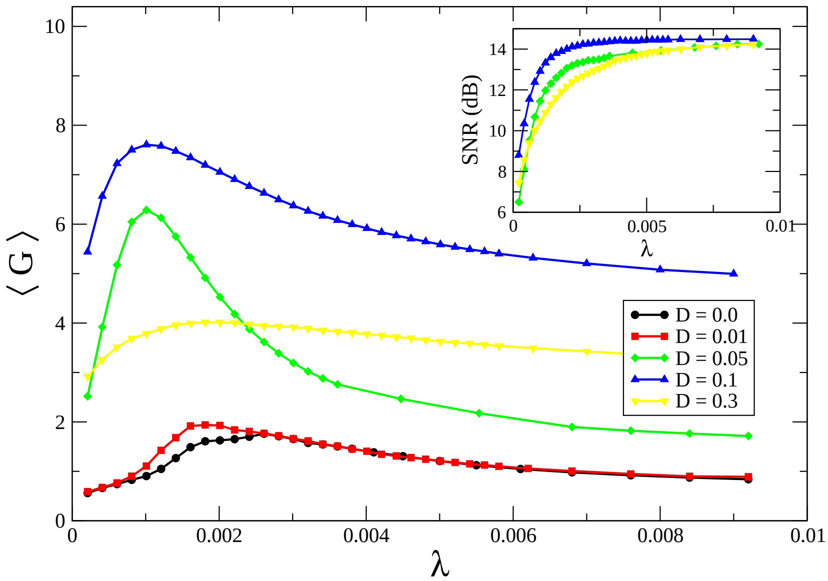

Fig. 2 Average amplification $\langle G \rangle$ as function of the coupling λ for four different values of the noise strength. Inset: The SNR output for the hub in the star-like network. Parameters are $A = 0.1$, $\omega = 2\pi \times 10^{-2}$.

Figure 2 illustrates how the amplification and SNR curves changes as the noise increases. The amplification grows initially with increasing noise until reaching a maximum value, decreasing for larger values of the noise. A small amount on noise facilitates the hub to surmount the potential barrier, mainly because the reamplified signal coming from the leaves now increases. This can be explained resorting to the dynamics of the leaves in presence of noise. As a first approximation this corresponds to the motion of a noisy overdamped oscillator in a bistable potential, whose dynamics is well understood in the framework of the stochastic resonance [2]. There, the response of the system to a deterministic signal attains a maximum value for a nonzero value of the noise signal. Far above such a value the amplification response decreases when the noise increases. This affects the average amplification for the hub, in that it decreases correspondingly. In addition, Fig. 2 shows that the amplification curves peak for lower coupling values for the same reason explained above. As for the SNR, it grows monotonically with increasing coupling, and noise values.

In Fig. 3, the average amplification as well as the SNR as function of the noise is plotted for several values of the coupling strength. Note that the amplification achieves a maximum value for $\lambda = 0.001$, decreasing for larger values, while for the SNR it grows monotonically with the coupling when the noise is sufficiently large. Therefore, for a given value of the noise, increasing the coupling improves the signal response of the system in terms of SNR. The explanation rests on the fact that increasing the coupling reduces the potential barrier, and thus facilitates the appearance of the characteristic SNR found in stochastic resonance phenomena.

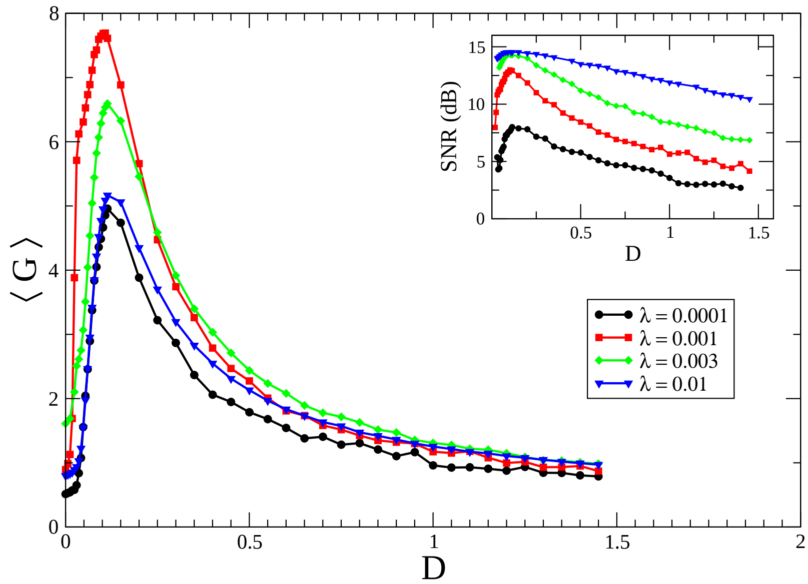

Fig. 3 Average amplification $\langle G \rangle$ as function of noise D for four different values of the coupling. Inset: The SNR output for the hub in the star-like network. Parameters are $A = 0.1$, $\omega = 2\pi \times 10^{-2}$.

6 Conclusions

We have extended the topological resonant-like effect found in [9] now in presence of external noise. Further, we have shown that an enhanced topological resonant-like effect can be induced by the optimization of the noise. The interplay between the cooperative interaction of nodes connected to the hub due to the topology, and the beneficial effects of the noise in surmounting the potential barrier, play a crucial role in the amplification of external signals. Moreover, an adjustment of the coupling can be used to improve further the system response, and may be important in the design of accurate remote sensing arrays.

We acknowledge support from the Spanish Ministry of Science and Technology, Grant FIS2006-13321-C02, and J.A.A. from the Ramon y Cajal programme. The assistance and usage of the resources provided by the BSC-CNS supercomputing facility is well appreciated.

References

1. Bulsara, A. R., Gammaitoni, L.: Tuning in to Noise. Phys. Today **49**, 39 (1996).
2. Gammaitoni, L., Hänggi, P., Jung, P., Marchesoni, F.:Stochastic resonance. Rev. Mod. Phys. **70** 223 (1998).
3. Lindner, J.F., Meadows, B.K., Ditto, W., Inchiosa, M.E., Bulsara, A.R.: Array Enhanced Stochastic Resonance and Spatiotemporal Synchronization. Phys. Rev. Lett. **75** 3 (1995).
4. Bollobas, B.: Random Graphs (2nd edition), Cambridge University Press, New York (2001).
5. Barabasi, A.L., Albert, R.: Emergence of scaling in random networks. Science **286**, 509–511 (1999).
6. Tessone, C.J., Mirasso, C.R., Toral, R., Gunton, J.D.: Diversity-induced resonance. Phys. Rev. Lett. **97** 194101 (2006)
7. Acebrón, J.A., Bonilla, L.L., Perez Vicente, C.J., Ritort, F., Spigler, R.: The Kuramoto model: A simple paradigm for synchronization phenomena. Rev. Mod. Phys. **77**, 137 (2005).
8. Gómez-Gardeñes, J., Moreno, Y., Arenas, A.: Paths to Synchronization on Complex Networks. Phys. Rev. Lett. **98**, 034101 (2007)
9. Acebrón, J.A., Lozano, S., and Arenas, A.:Amplified signal response in scale-free networks by collaborative signaling. Phys. Rev. Lett., in press (2007).