Phase Patterns of Coupled Oscillators with Application to Wireless Communication

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Abstract. Here we study the plausibility of a phase oscillators dynamical model for time division for multiple access in wireless communication networks. We show that emerging patterns of phase locking states between oscillators can eventually oscillate in a round-robin schedule, in a similar way to models of pulse coupled oscillators designed to this end. The results open the door for new communication protocols in a continuous interacting networks of wireless communication devices.

Keywords: time division for multiple access, phase oscillators, round-robin schedule.

1 Introduction

Nowadays, wireless communications have become pervasive. This form of telecommunication between elements forming a network has technically evolved to the third generation of wireless systems, that incorporates the features provided by broadband. With this evolution, wireless networks become a plausible candidate for the main telecommunication mechanism in the next future. At the same time, this technical advance comes along with new problems which requires the use of innovative ideas to solve them. One of the problems we are aware, is that of maintaining decongestion in single-hop networks, where time division for multiple access (TDMA) strategies have been shown to be a good scheme for message transmission [1]. TDMA is a channel access method for shared medium (usually radio) networks. It allows several users to share the same frequency channel by dividing the signal into different timeslots. The users transmit in rapid succession, one after the other, each using his own timeslot. This allows multiple stations to share the same transmission medium (e.g. radio frequency channel) while using only the part of the bandwidth they require.

Between the algorithms that have been proposed to solve this problem, a bio-inspired solution called *desynchronization*, has attracted our attention [2,3]. The idea is to mimic some synchronization processes in biological systems, modeled by pulse-coupled oscillators in networks. Within this scenario a mapping between

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wireless nodes in a network and pulse-coupled oscillators is possible, providing a simple and elegant protocol for TDMA communication. Here, desynchronization refers specifically to a state where nodes perfectly interleave periodic events to occur in a round-robin schedule, in contrast with synchronization where all the oscillators collapse their phase behavior.

The model presented in Ref. [2] recalls some results in lattices of coupled oscillators where spatio-temporal pattern form in a ring of oscillators with inhibitory unidirectional pulse-like interactions [4,5] inspired in the behavior of elementary neural systems. The attractors of the dynamics are limit cycles where each oscillator fires once and only once in a cycle, and some of them correspond to the desired behavior of round-robin schedule, that maintain order in the firing succession. The limit cycle structure of the attractors of pulse-coupled oscillators system shown in [4,5], is akin to those limit cycle emergent in coupled phase oscillators, in particular in the Kuramoto model [6,7]. Using this similarity on the final states, between both descriptions, here we study the plausibility of a self-organized algorithm between nodes communicating in a wireless network, using the dynamics of phase oscillators. The results show that an equivalent desynchronized state is obtained for a finite set of initial conditions, in a continuous interacting model. A reseting mechanism is proposed to account for the entry and exit of nodes of the network, while maintaining the desynchronized state.

The paper is organized as follows. In Sect. II we review some of the results known for a ring of pulse-coupled oscillators. In Sect. III we introduce the general model of phase-coupled oscillators and in the next section we present our proposal of phase oscillators continuously coupled through a function that depends on the sinus of the phase difference of the two interacting (neighboring) units. In Sect. V we analyze the stability of the fixed points of the collective dynamics, whereas in Sect. VI we study the effect of adding or removing nodes to our system, maintaining the prescribed schedule. Finally, in the last section, we present a brief discussion of the results and provide some lines of future research.

2 Patterns in Pulse-Coupled Oscillators in a Ring

Generally speaking, coupled oscillators interact via mutual adjustment of their amplitudes and phases. When coupling is weak, amplitudes are relatively constant and the interactions could be described by phase models. In particular, pulse-coupled oscillators account for some biological processes like heart pacemaker cells, integrate and fire neurons, and other systems made of excitable units. The instantaneous interactions that take place in a very specific moment of its period makes the treatment of these systems more complicated from a theoretical point of view. In any case, the richness of behaviors os these pulse-coupled oscillatory systems include synchronization phenomena, spatio-temporal pattern formation (traveling waves, chess-board structures, and periodic waves), rhythm annihilation, self-organized criticality, etc. The reader is pointed to Ref. [5] for references on this subject.

In these models, the phase of each oscillator evolves linearly in time – usually all units having the same period. When reaching some precise value of the phase the oscillator fires emitting a pulse that is received instantaneously by its set of neighbors. At this point the neighbors change their phases according to some specific function, called phase response curve. One should notice that this response function plays the crucial role in the dynamics of the population. Since two different time scales are in play, a continuous description makes no sense and the usual way to describe mathematically the system is by means of maps. A map represents the total evolution of driving (independent linear evolution in the slow time scale) and firing (interaction between units through a pulse) processes and the change in state after a complete map reflects the nature of the dynamical behavior. Thus, we can observe the evolution towards the attractors and analyze the stability of the fixed points.

In particular, in a set of works by the authors of the current paper, it was theoretically analyzed the behavior of rings of oscillators subjected to a linear phase response curve. In the first work [4] we dealt with unidirectional couplings in the ring obtaining exact values of the fixed points of the dynamics. As we said before, the stability of the fixed points is given by the return maps of the driving plus firing process. We computed the bounds of the eigenvalues of the matrix that describes the map and showed that any excitatory coupling (positive linear phase response curve) has unstable fixed points and the only solution is a synchronized state in which the oscillators collapse one by one. On the other hand, for an inhibitory coupling (negative linear phase response curve) the fixed points become stable, giving rise to spatio-temporal patterns where a constant phase-difference between oscillators is achieved. In a second work [8] we extended the previous result to a population of bidirectionally coupled units. Finally, in a third work [5] we analyze in much more detail the patterns that appear for inhibitory couplings (negative phase response curve). In particular, we were able to find the probability of selecting a given pattern under arbitrary initial conditions. In a ring of N oscillators there are $(N-1)!$ possible permutations of the firing sequence, by keeping one of the oscillators as the initial firing one. But all these possible sequences can give rise a smaller number of fixed points, which is $N-1$. Then these fixed points or patterns have some degeneracy that can be computed analytically. From this degeneracy, it can be computed the probability of pattern selection, that depends also on the coupling strength. For instance, it can be easily found that, in the case of small coupling, the most probable state is that with the maximum phase difference between neighbors, i.e. the phase-opposition (antisynchronization) state, and as we increase the number of oscillators the patterns distribution gets sharpened around this value. There is an additional effect in the pattern selection for this construction. When the coupling strength increases there are some fixed points that disappear, i.e. there are no longer part of the available configuration space. Depending on the number of oscillators and on the periodicity of the patterns, we could estimate the critical value of the coupling strength for which the pattern disappears. This effect is, of course, very important since it alters the distribution of the pattern selection.

3 Coupled Phase Oscillators

In contrast to pulse -coupled oscillators, phase coupled oscillators are described in a single time scale by a driving term plus an interaction between their (usually relatives) phases

$$\dot{\varphi}_i(t) = f_i(\varphi_i(t)) + \sigma \sum_j g(\varphi_i(t), \varphi_j(t)) \tag{1}$$

where φ stands for the phases, σ for the coupling strength, and f and g for general functions of the specified arguments. The sum runs over the neighboring units of oscillator i.

The behavior of 1D lattices of phase models is considerably complex, even for nearest neighbor coupling. In the case of chains of oscillators, for example, when coupling is local, oscillators at the ends get different inputs from those in the middle so that phase locking may not even exist. As long as the differences in the frequencies are small enough, there will be a phase-locked solution. Interestingly, nearest neighbor interaction chains can support very small gradients when the coupling term has the form of the sinus of the phase difference (and, in fact, any odd periodic function). However, if the coupling function contains even components (that is, replace $\sin(\varphi)$ with $\sin(\varphi+\delta)$), then frequency gradients as that are can be supported in nearest neighbor chains of coupled phase oscillators [9,10].

One of the most useful connections between the description of pulse-coupled oscillators and phase oscillators is exploited in the so-called transformation to phase models [11]. Fulfilling the condition of weak coupling, and autonomous oscillatory behavior of the pulse oscillators, the entire network can be transformed into a simpler phase model by a piece-wise continuous change of variables. The interest of this mathematical equivalence is that many pulse-coupled systems can be viewed as phase-coupled systems whose continuous description is more amenable. Driven by this analogy we explore the performance of a simple set of phase oscillators in a ring compared to the use of pulse-coupled oscillators described in [2], for TDMA on wireless networks.

4 The Kuramoto Model in a Locally-Coupled Ring

We consider here a particular model of phase oscillators that was introduced by Kuramoto [6]. In the original paper, Kuramoto analyzed a population of oscillators with an all-to-all pattern of connectivity. In principle, each oscillator has its own frequency drawn from a random distribution and is coupled via a sine function to the rest of the population

$$\frac{d\varphi_i}{dt} = \omega_i + \sigma \sum_j \sin(\varphi_j - \varphi_i) \qquad i = 1, ..., N \tag{2}$$

In most of the analysis the interesting issue has been the transition to the synchronized state that appears above some critical value of the coupling strength

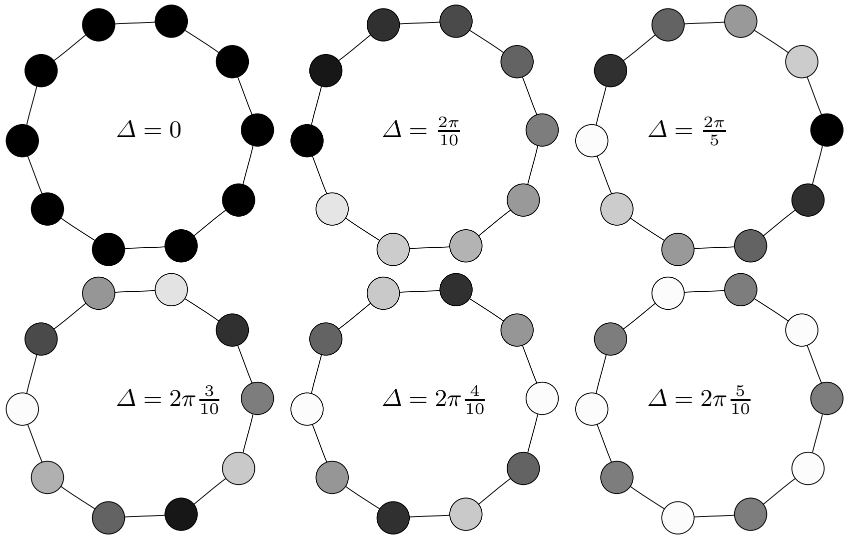

Fig. 1. Stable configurations for a ring of 10 oscillators. Top: for $\delta = 0$. Bottom: for $\delta = \pi$. The color of the node stands for the phases (white for $\varphi = 0$ to black for $\varphi = 2\pi$), but only the phase difference between neighboring nodes is important. The label at the center of each graph stands for the phase difference between neighboring nodes.

σ [7]. However, our goal here is to analyze the stationary states that can appear for a particular topology of the connections when all oscillators are driven by the same frequency (that can be taken as 0 without loss of generality). Hence, as the simples configuration, we consider a 1D ring of N oscillators, where each unit is connected to its nearest neighbors, and have zero inner frequency

$$\dot{\varphi}_i = \sin(\varphi_{i+1} - \varphi_i + \delta) + \sin(\varphi_{i-1} - \varphi_i + \delta) \qquad \forall i = 1, ...N. \tag{3}$$

Here we have also introduced an arbitrary phase shift δ that will play a key role when considering the symmetries of the final stationary state.

A stationary solution $\varphi_{i+1} - \varphi_i = \Delta$, $\forall i$ exists provided that $\Delta = 2\pi m/N$, being $m \in \mathbb{N}$.[1] In this case we should have for all the oscillators

$$\dot{\varphi}_i = 2 \sin \delta \cos \Delta \tag{4}$$

i.e. the oscillators rotate at this effective frequency, all with the same value but a fixed phase difference between neighbors is kept. In Fig. 1 we plot a ring of 10 oscillators for a stationary phase difference corresponding to the cases $m = 0, 1, 2, 3, 4, 5$ (the remaining cases correspond to the complementary ones to these, because all results have to be understood as mod 2π).

[1] Notice that the cases m and $N - m$ are equivalent since it is a positive or a negative phase difference and all phase differences are to be understood mod 2π.

5 Linear Stability of the Attractors

Let us assume a small instantaneous perturbation to one of the nodes: $\varphi_i \rightarrow \varphi_i + \varepsilon$. Then the equation of motion for this oscillator becomes

$$\dot{\varphi}_i = \sin(\Delta + \delta - \varepsilon) + \sin(-\Delta + \delta - \varepsilon). \tag{5}$$

Expanding the sinus functions we get up to linear order in ε

$$\dot{\varphi}_i = 2 \sin\delta \cos\Delta - 2\varepsilon \cos\delta \cos\Delta. \tag{6}$$

The derivative of the frequency with respect to the perturbation is $d\dot{\varphi}_i/d\epsilon = -2\cos\delta\cos\Delta$, providing the stability of the stationary solutions of the system. Let us now look in detail to the different combinations of these terms, keeping in mind that δ is a prescribed phase that breaks the symmetry of the problem. We will consider only two cases ($\delta = 0$ and $\delta = \pi$), any case in between these values only affects the effective frequency. Notice, however, that $\delta = \pi/2$ is a very particular case and the stability analysis requires a specific study that is beyond the scope of the current work.

For the case $\delta = 0$ all states with $\cos(\Delta) > 0$ are stable, i.e. $d\dot{\varphi}_i/d\epsilon < 0$, in particular the synchronized state $\Delta = 0$. But there are also other possibilities $0 < m < N/4$ for which the oscillators can end in a stable stationary state. Notice that the case $m = 1$, which is a stable solution whenever $N \leq 4$, corresponds to the minimum phase difference between oscillators, that is the case of the round-robin schedule mentioned in the Introduction. In Fig. 1 (top) we show the three stable configurations for a ring of 10 oscillators with $\delta = 0$.

For the case $\delta = \pi$, new stable states appear, all those with $N/4 < m \leq N/2$, and the synchronized state becomes unstable. For the particular case of 10 oscillators, we show the three stable configurations in the bottom of Fig. 1.

In general, we obtain a set of stable configurations where there can be subsets of nodes which are partially synchronized. For instance, if N is even, there always exists a configuration, stable for $\delta = \pi$, for which the phase difference between any two neighboring nodes is π and hence we have some sort of local *antisynchronization*, which is the maximum phase difference between neighboring oscillators. Another interesting case is that of N being a prime number; in this case all stable configurations are equivalent in the sense that there are no two synchronized oscillators, and the round-robin schedule is maintained, although not necessarily for neighboring nodes. Any stable configuration establishes a different order for the evolution of the oscillators. Although in the case of phase-coupled oscillators the firing does not make any sense, it is important to specify some value of the phase, for instance its maximum value 2π. Then any configuration stands for the time sequence of the oscillators phases reaching the value 2π.

The persistence of stable configurations where a round-robin scheduled is satisfied, opens the door for a self-organized solution to the problem of TDMA in wireless networks. However, there is still a problem concerning the inclusion of new agents (oscillators) to the system. In the next section, we investigate the effect of such new incorporations to the existing system.

6 Variation on the Number of Nodes in the Network

First of all we are going to consider how the round-robin stationary state responds to the addition of a new oscillator. Then we have, as starting configuration, a set of $N-1$ nodes such that

$$\varphi_i = i * (2\pi)/(N-1) \ \ \forall i = 1, \ldots, (N-1) \tag{7}$$

and now we add an incoming oscillator, that we label N. The phase of the incoming oscillator is unknown. For this reason, as a first approximation, we discretize the possible values of the incoming phase φ_N in the range $[0, 2\pi]$ and count the fraction of values of this set that leads to a new round-robin configuration of N oscillators. In Fig. 2 we plot the fraction of values of the initial phase φ_n that give rise either to the round-robin state, with N oscillators, or to the synchronized state. We have not observed the emergence of other states, although at this point we can not discard their existence as spurious states.

We notice that the round-robin state is very robust in the sense that it emerges from the new configuration almost surely, although the time response is large and it increases with the number of oscillators.

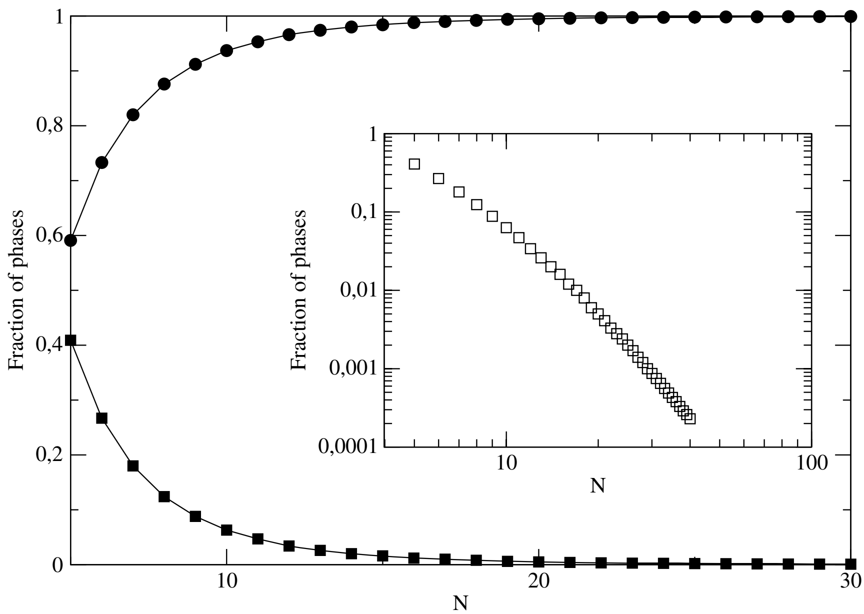

Fig. 2. Fraction of values of the initial phase of the incoming oscillator that leads to a round-robin configuration (top) and to a synchronized state (bottom), as a function of the number of final nodes. In the inset we plot the fraction of configurations that give rise to the synchronized state in log-log scale, to how fast it decreases with the system size.

On the other hand, the fact of removing one node from a stable configuration is also quite robust. The round-robin state is broken exclusively for a very small number of oscillators. This is a deterministic case in which we fix all initial phases such that $\Delta = 1/(N+1)$ and the system evolves deterministically towards $\Delta = 1/N$ for N larger than 6.

7 Discussion

We have presented a bio-inspired approach to the round-robin schedule of wireless networks, based on the synchronization of phase oscillators, particularly, Kuramoto oscillators in a ring with nearest neighbors coupling. The study of patterns of phase locking attractors shows that this continuous interaction model could be also used as an alternative protocol for TDMA, comparable to the approach of pulse coupled oscillators presented in [2], although still to be developed in deep. Upcoming experiments in complex topologies of phase oscillators show that the number of possible phase locking patterns is hugely rich. We will present a systematic study of the different stationary states, as well as their relative basis of attraction in different topologies in a future work.

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