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Phase Locking in a Network of Neural Oscillators.

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Abstract. – We present an analytical approach that allows to compute the long-time behaviour of networks with oscillatory behaviour. We show that phase locking is a mechanism to store information in the system. This technique is an interesting alternative to conventional methods of analysis of associative memories.

The analysis of dynamical properties of attractor neural networks (ANN) has been the focus of important works in the last few years [1,2]. It provides information about the short- and long-time behaviour of networks characterized by symmetric and asymmetric couplings and also allows to understand the nature of some collective phenomena, such as mutual synchronization in the temporal activity of large assemblies of neurons, which are responsible for interesting effects related to the processing of information observed in real experiments [3].

To reproduce synchronization between members of a population it is important to introduce elements which could take into account the degree of coherence in the temporal response of active neurons. The conventional models of ANN characterizing the activity of the neurons through binary values [4] are not suitable for this task and more complex descriptions are necessary. Abbott [5] and Schuster *et al.* [6] have shown that some biologically motivated models capable of accounting for the oscillatory behaviour of neurons can be written in terms of phase equations after a suitable transformation. This is quite interesting because the new simplified description allows an analytical treatment. Additionally, these models may present typical properties of associative memory, not in terms of fixed points but in terms of phase locking. However, the techniques used to analyse their long-time behaviour are essentially qualitative. Our goal is to present an approach which allows to get rigorous results in the stationary state.

Our starting point is one of the best-known models of phase oscillators, the so-called Kuramoto's model [7]. Several versions of it have appeared in other studies about synchronization in neural systems [5,6,8]. In this model the phase of each element of the population evolves according to the following Langevin equation:

$$\frac{d\theta_i}{dt} = \omega_i + \gamma_i(t) + \sum_{j=1}^{N} J_{ij} \sin(\theta_j - \theta_i), \tag{1}$$

where J_{ij} is the coupling matrix, θ_i and ω_i are the phase and the natural frequency of the i-th oscillator, ω_i is picked up randomly from a certain distribution $g(\omega)$, N is the size of the population and $\gamma_i(t)$ are independent white-noise random processes with zero mean and correlation

$$\langle \gamma_i(t) \gamma_j(t') \rangle = 2D\delta_{ij}\delta(t - t'), \qquad D \geqslant 0. \tag{2}$$

The specific form proposed for the couplings is the bridge that allows to make an analogy between models of phase oscillators and ANN. We have considered a population of N neurons, active at high rate during a given period of time and presenting an oscillatory behaviour. We have also assumed that the synaptic efficacies may contain information about the phase of each element. As usual in ANN we want to store p sets of random patterns (phases) $\{\xi\}$ and a simply way to do this is to assume that

$$J_{ij} = \frac{J}{N} \sum_{\mu=1}^{p} \cos(\xi_i^\mu - \xi_j^\mu), \tag{3}$$

where J is the intensity of the coupling. This form preserves the basic idea of Hebb's rule but now adapted to the symmetry of our problem. Our goal is to determine the stationary properties of the model described by eqs. (1) and (3) through a mean-field formalism widely used in the analysis of large populations of coupled oscillators [9], but new in the treatment of ANN. Notice that when the distribution of frequencies vanishes ($g(\omega) = \delta(\omega)$) our neurons are no longer oscillators. In this case our system becomes a Q-state clock model of neural network in the limit $Q \to \infty$. It has strong analogies with a model studied by Cook [10]. We will show that with the method discussed in this paper it is possible to reproduce the results of [10] in a different way, emphasizing the relevant influence of $g(\omega)$ on the long-time properties of the system.

To analyse our model, it is convenient to introduce the following order parameters:

$$q_\pm^\mu \exp[i\phi_\pm^\mu] = \frac{1}{N} \sum_j \exp[i(\theta_j \pm \xi_j^\mu)]; \tag{4}$$

$\phi_\pm^\mu$ play the role of a mean phase, q_-^μ measures the correlation between the state of the system and the pattern ξ^μ, and q_+^μ is another correlation not relevant in our study. Notice that the state of the system, described by an N-dimensional vector whose i-th component is the phase of the i-th oscillator, is changing continuously in time. This means that except for $\omega \approx 0$ there are not fixed points of the dynamics. However, as we have mentioned previously this is not a problem since it is possible to store information as a difference of phases between pairs of oscillators (this fact justifies the choice of the learning rule (3)), a quantity that may remain constant in time. Therefore, if the initial state is correlated (in terms of phase locking) with one of the embedded patterns, the final state will also have a macroscopic correlation with the same pattern provided p is below a critical value. Then, the interesting physical quantity which can describe synchronization effects is q^2, since

$$q^2 = \frac{1}{N^2} \sum_i \sum_j \exp[\theta_i - \xi_i - \theta_j + \xi_j] = \langle \cos(\theta - \xi) \rangle^2 + \langle \sin(\theta - \xi) \rangle^2. \tag{5}$$

Now, the evolution equation for the phase oscillators is

$$\frac{d\theta_i}{dt} = \omega_i + \frac{J}{2}\sum_{\mu=1}^{p}[q_-^\mu \sin(\phi_-^\mu - \theta_i + \xi_i^\mu) + q_+^\mu \sin(\phi_+^\mu - \theta_i - \xi_i^\mu)] + \gamma_i(t). \tag{6}$$

The main idea is to realize that in the thermodynamic limit $N \to \infty$, it is possible to derive a non-linear Fokker-Planck equation for the one-oscillator probability density $\rho(\theta, t, \omega, \xi)$ [11]

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\theta}[\mathcal{D}_v\rho] - D\frac{\partial^2\rho}{\partial\theta^2} = 0, \tag{7}$$

where $\mathcal{D}_v$ is the drift velocity term

$$\mathcal{D}_v = \left[\omega + \frac{J}{2}\sum_{\mu=1}^{p}[q_+^\mu \sin(\phi_+^\mu - \theta - \xi^\mu) + q_-^\mu \sin(\phi_-^\mu - \theta + \xi^\mu)]\right]. \tag{8}$$

Now, the order parameters (4) become

$$q_\pm^\mu \exp[i\phi_\pm^\mu] = \int \dots \int d\omega\, d\theta \exp[i(\theta \pm \xi^\mu)]\rho(\theta, \omega, \xi^1, \dots, \xi^p) g(\omega) \prod_{\mu=1}^{p} f_\mu(\xi^\mu)\, d\xi^\mu, \tag{9}$$

where $g(\omega)$ and $f_\mu(\xi^\mu)$ are the frequency and pattern distribution, respectively. Since we are interested in the long-time behaviour of the system, we have solved eq. (7) by imposing stationary conditions leading to

$$\rho(\theta, \omega, \xi^\mu) = \frac{M(\theta)\int_0^{2\pi} d\eta\, N(\theta, \eta)}{\mathcal{Z}}, \tag{10}$$

where

$$M(\theta) = \exp\left[\frac{J}{2D}\sum_{\mu=1}^{p}[q_+^\mu \cos(\phi_+^\mu - \theta - \xi^\mu) + q_-^\mu \cos(\phi_-^\mu - \theta + \xi^\mu)]\right], \tag{11}$$

$$N(\theta, \eta) = \exp\left[-\frac{\omega\eta}{D} - \frac{J}{2D}\sum_{\mu=1}^{p}[q_+^\mu \cos(\phi_+^\mu - \theta - \eta - \xi^\mu) + q_-^\mu \cos(\phi_-^\mu - \theta - \eta + \xi^\mu)]\right] \tag{12}$$

and

$$\mathcal{Z} = \int_0^{2\pi} d\theta\, M(\theta)\int_0^{2\pi} N(\theta, \eta)\, d\eta. \tag{13}$$

Equations (9)-(12) describe the behaviour of the system in the most general case. However, in this study we have only considered the low loading limit, *i.e.* when the capacity $\alpha = p/N$, defined as the ratio between the number of patterns and the number of units of the system, goes to zero. In this limit we can assume that when the initial state of the system has a macroscopic correlation with a pattern μ, only the order parameter $q_-^\mu \equiv q$ will be relevant, which simplifies notably the nature of the problem. The situation with $\alpha \neq 0$ will be considered elsewhere.

To calculate q we can proceed in two different manners, either by solving directly the equation, which is complex because it means to solve an integral equation implying to get values of q through numerical integration, or by identifying $\mathcal{Z}$ as a generating functional of the order parameters. This method is more elegant and gives algebraic expressions easier to deal with. Let us rewrite $\mathcal{Z}$ as

$$\mathcal{Z} = \int_0^{2\pi} \mathrm{d}\theta\, M(\theta, \sigma, \Phi) \int_0^{2\pi} N(\theta, \eta)\, \mathrm{d}\eta\,, \tag{14}$$

where

$$M(\theta, \sigma, \Phi) = \exp[\sigma \cos(\Phi - \theta + \xi)]\,, \tag{15}$$

then it is straightforward to see that

$$q = \left\langle\!\!\left\langle \frac{\partial}{\partial \sigma} \ln \mathcal{Z} \,\Big|_{\sigma = Jq/2D,\, \Phi = \phi} \right\rangle\!\!\right\rangle, \tag{16}$$

where $\langle\!\langle ... \rangle\!\rangle$ is an average over ω and ξ. Integrating (13), averaging over ξ and evaluating the partial derivative (16) we obtain a self-consistent equation for the q parameter:

$$q = \left\langle \frac{\dfrac{D}{\omega} I_0(\beta q) I_1(\beta q) + \sum\limits_1^{\infty} \dfrac{(-1)^n}{(\omega/D)^2 + n^2} I_n(\beta q)(I_{n-1}(\beta q) + I_{n+1}(\beta q)) \left(\dfrac{\omega}{D}\right)}{\dfrac{D}{\omega} I_0^2(\beta q) + 2 \sum\limits_1^{\infty} \dfrac{(-1)^n}{(\omega/D)^2 + n^2} I_n^2(\beta q) \left(\dfrac{\omega}{D}\right)} \right\rangle_{\omega}, \tag{17}$$

where I_n are the modified Bessel functions of first kind of order n, $\beta = J/2D$ and $\langle\ \rangle_\omega$ means an average over the distribution of frequencies. Taking into account the symmetry properties of the modified Bessel functions for n integer ($I_n(x) = I_{-n}(x)$), we can summarize this formula in

$$q = \left\langle \frac{\sum\limits_{-\infty}^{\infty} \dfrac{(-1)^n}{\omega^2 + D^2 n^2} I_n(\beta q) I_{n-1}(\beta q)}{\sum\limits_{-\infty}^{\infty} \dfrac{(-1)^n}{\omega^2 + D^2 n^2} I_n^2(\beta q)} \right\rangle_{\omega}. \tag{18}$$

In practice the numerical computation of this algebraic expression is not difficult because the maximum contribution to the series comes from the modified Bessel functions of lower orders. In contrast with conventional models of ANN for which a positive overlap is always found below the critical temperature (in the limit $\alpha \to 0$), here phase locking can be destroyed if the distribution of frequencies is sufficiently broad. From a linear analysis of (16) it is straightforward to show that, as

$$\int_{-\infty}^{\infty} \frac{g(\omega)}{(\omega^2/D^2) + 1}\, \mathrm{d}\omega < \beta^{-1}\,, \tag{19}$$

no synchronization is allowed.

Finally, it is interesting to compare our results with those given by Cook in [10] in the limit of $Q \to \infty$. We observe from (18) that when $\omega \to 0$ (absence of frequencies) the order parameter q is

$$q = \frac{I_1(\beta q)}{I_0(\beta q)}, \tag{20}$$

which is exactly the same expression reported by Cook, showing that for $\alpha \to 0$ both models behave identically, although this is not true for finite α. However, our result has been derived in a more general context, since we have included the effect of a distribution of frequencies and additionally it is not difficult to deal with more complex situations (*e.g.* external fields).

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