Physica A 201 (1993) 614–625
North-Holland

SDI: 0378-4371(93)E0340-K

PHYSICA A

Phase diagram of a planar XY model with random field

A. Arenas and C.J. Pérez Vicente
Departament de Física Fonamental, Facultat de Física, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain

Received 7 April 1993

We have studied a infinite-range planar XY model in the presence of a random field. We have analyzed the dynamics of the system through a Fokker–Planck formalism, focussing our attention in the long time behaviour. This technique allows to compute exactly the phase diagram of the model.

1. Introduction

The effect of random fields and anisotropy on the critical properties of magnetic systems has been a matter of study in the last years. It is well known [1] that below $d=4$ a weak random field destroys long-range order in systems with continuous symmetry. In particular, for the planar XY model several authors [2–5] have investigated the critical behaviour of low dimensional systems through different theoretical approaches such as renormalization group theory, field theory methods or series of exact transformations. Of particular interest for us is the work of Goldschmidt and Schaub [6] who study the problem via time dependent Langevin formulation. This formalism allows to average over the disorder at the outset of the calculation and to avoid the inconvenients emerged from the use of the replica method. More recent studies still investigate the effects of external magnetic fields but now in systems with complex topology such as 3D anisotropic XY models.

The analogous problem in models with long-range interactions has not been analyzed so deeply. In this situation a random field does not necessarily destroy long-range ferromagnetic order but still introduces interesting effects on the critical properties of systems with continuous symmetry. The most relevant studies in this area have been performed for systems with quenched random bonds such as vector spin glasses [7,8] in presence of anisotropy or an uniaxial

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constant external field. However, the effect of a random distribution of fields (site disorder) is not so well studied even for ferromagnetic couplings.

In this paper we solve a mean-field planar XY model in the presence of a random field. Our goal is to study the stationary properties of the model, but neither through the usual techniques of equilibrium statistical mechanics, nor other formalisms that allows to compute, in terms of correlation and response functions, the dynamic evolution of the system for all time. Instead of this, we have analyzed the dynamics of the system through a Fokker–Planck formalism, quite usual in the treatment of non-equilibrium problems. As will show this powerful technique turns out to be an excellent alternative to standard procedures to tackle problems with quenched disorder. We will derive a set of selfconsistent equations that allows to compute exactly the phase diagram of the model for an arbitrary distribution of fields.

The paper is structured as follows: section 2 is devoted to the description of the model discussing their analogies with other physical systems. In section 3 we solve the dynamics in the stationary state leading to analytical results which are discussed and compared with simulations in section 4. Finally two appendixes will give information about technical details.

2. The model

We have considered a planar XY model described by the Hamiltonian

$$H = -\sum_{i,j} J_{ij} \cos(\theta_j - \theta_i) + \sum_i h_i \cos\theta_i \,, \tag{1}$$

where $J_{ij} = J/N$ is a mean-field ferromagnetic interaction between all the N spins and h_i is the random field distributed over the population of spins with an arbitrary probability density $g(h)$. It describes the effect of a quenched random distribution of impurities. In our study we have considered a simplified situation in which all the h_i have the same orientation and only change in modulus. The case where h_i is a random p-fold symmetry breaking field is technically more complex and will not be analyzed in this paper although it can be studied along the same lines sketched later. In addition, we have assumed a relaxational dynamics for the variables θ_i

$$\frac{\partial \theta_i}{\partial t} = -\frac{\partial H}{\partial \theta_i} + \eta_i(t) \,, \tag{2}$$

where η_i is a Gaussian white random process with zero mean and correlation

$$\langle \eta_i(t)\,\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t')\,, \qquad D \geq 0\,. \tag{3}$$

After these considerations eq. (2) reads

$$\frac{\partial\theta_i}{\partial t} = \frac{J}{N}\sum_j \sin(\theta_j-\theta_i) + h_i \sin\theta_i + \eta_i(t)\,. \tag{4}$$

Notice that the model described by (4) is a particular case (a 'static' version) of a more general dynamic system made of active plane rotators. In this new situation each spin rotates with a natural frequency ω_i randomly distributed with $f(\omega)$. Now, the phase evolves in time following

$$\frac{\partial\theta_i}{\partial t} = \omega_i + \frac{J}{N}\sum_j \sin(\theta_j-\theta_i) + h_i \sin\theta_i + \eta_i(t)\,. \tag{5}$$

This evolution equation appears also in a different context. It has been used to study effects of synchronization [9] in the temporal activity of members (modelled as phase oscillators) of a population. This phenomenon has been observed frequently in biological systems. In absence of external fields it is known that for weak couplings the oscillators run at its own frequency whereas beyond a critical J_c/D synchronization appears spontaneously. Shinomoto et al. [10,11] and Sakaguchi [12] have also studied this model in presence of a constant external force. However, their approach do not consider any kind of disorder, is essentially numerical and far from the analytical exact mean-field solution presented in this paper. We have remarked this analogy because the long time behaviour of the planar XY model will be analyzed with the same technique usually applied to the study of the dynamic properties of systems of phase oscillators. In this sense, the methodology discussed in this paper is an alternative to standard methods of equilibrium statistical mechanics.

In order to describe mathematically the collective behaviour of the system it is convenient to introduce the following order parameters:

$$m\,\mathrm{e}^{\mathrm{i}\phi} = \frac{1}{N}\sum_j \mathrm{e}^{\mathrm{i}\theta_j}\,, \tag{6}$$

m gives information about the degree of coherence (magnetization) of the population and ϕ is a mean phase (mean orientation). In terms of these parameters the dynamic evolution of each element (for the most general case) is given by the following Langevin equation:

$$\frac{\partial\theta_i}{\partial t} = \omega_i + Jm\sin(\phi-\theta_i) + h_i\sin\theta_i + \eta_i(t)\,. \tag{7}$$

In the limit of $N \to \infty$ it is possible to derive a nonlinear Fokker–Planck equation for the one oscillator probability density. $\rho(\theta,t,\omega,h)\,\mathrm{d}\theta$ gives the fraction of spins (rotators) of natural frequency ω that 'feel' a field h lie between θ and $\theta + \mathrm{d}\theta$. The resulting expression for ρ is shown to obey [13]

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}(\rho v) , \tag{8}$$

where the drift velocity v is given by

$$v(\theta, t, \omega, h) = \omega + Jm \sin(\phi - \theta) + h \sin \theta . \tag{9}$$

Eq. (8) describes the evolution of the whole system for all time. However, we overlook transient effects and focus our interest on the long time behaviour, which is studied in the next section.

3. Analysis of the stationary state

Since our goal is to analyze the stationary behaviour of the system we consider solutions of the Fokker–Planck equation (8) with $\partial\rho/\partial t = 0$. Normalization and periodicity conditions lead to

$$\rho(\theta, \omega, h) = \frac{\exp[(Jm/D)\cos(\phi - \theta) - (h/D)\cos\theta] \int_0^{2\pi} \mathrm{d}\eta\, H(\theta, \eta)}{\mathcal{Z}} , \tag{10}$$

where

$$H(\theta, \eta) = \exp\Big(-\frac{\omega\eta}{D} - \frac{Jm}{D}\cos(\phi - \theta - \eta) + \frac{h}{D}\cos(\theta + \eta)\Big) , \tag{11}$$

and

$$\mathcal{Z} = \int_0^{2\pi} \exp\Big(\frac{Jm}{D}\cos(\phi - \theta) - \frac{h}{D}\cos\theta\Big)\,\mathrm{d}\theta \int_0^{2\pi} H(\theta, \eta, h)\,\mathrm{d}\eta . \tag{12}$$

A detailed mathematical derivation of these equations can be found in appendix A. Now let turn back to the static planar XY model, by considering no frequencies in (11). A complete study of the more general system will be presented elsewhere. The magnetization m and the mean orientation ϕ can be computed from ρ as

$$m=\int_0^{2\pi} e^{i(\theta-\phi)}\,\rho(\theta,h)\,g(h)\,d\theta\,dh \tag{13}$$

by equating real and imaginary parts of both sides and solving the respective integral equations simultaneously. This task is not easy from a technical point of view because we have to compute normalized integrals that form part of the integrand of other integrals. Fortunately there is another way to compute these quantities in a much simpler and clever manner. Due to the normalized structure of $\rho(\theta,h)$ it is convenient to define a generating functional $\mathscr{Z}$ as

$$\mathscr{Z}=\int_0^{2\pi} d\theta\,\exp\left(\sigma\cos(\psi-\theta)-\frac{h}{D}\cos\theta\right)\int_0^{2\pi} H(\theta,\eta)\,d\eta\,. \tag{14}$$

Taking into account that the magnetization must be real, it is straightforward to show that the order parameters can be derived from $\mathscr{Z}$ as

$$m=\int m(h)\,g(h)\,dh=\left\langle\!\!\left\langle\frac{\partial}{\partial\sigma}\ln\mathscr{Z}\big|_{\sigma=Jm/D,\psi=\phi}\right\rangle\!\!\right\rangle, \tag{15}$$

$$0=\left\langle\!\!\left\langle\frac{\partial}{\partial\psi}\ln\mathscr{Z}\big|_{\sigma=1,\psi=\phi}\right\rangle\!\!\right\rangle, \tag{16}$$

where $\langle\!\langle\cdots\rangle\!\rangle$ is an average over the distribution $g(h)$. This approach has several advantages since it allows to find algebraic equations for $m(h)$, what simplifies notably the numerical resolution of the resultant expressions and allows a more detailed analytical analysis of them. Integrals in (14) can be performed by developing the argument of the exponential in series of Bessel functions [14], leading to (see appendix B)

$$m=\int\frac{\sum_{n=-\infty}^{\infty}(-1)^n I_n(h/D)\,I_{n-1}(Jm/D)\cos(n\phi)}{\sum_{n=-\infty}^{\infty}(-1)^n I_n(h/D)\,I_n(Jm/D)\cos(n\phi)}\,g(h)\,dh\,, \tag{17}$$

$$0=\int\frac{\sum_{n=-\infty}^{\infty}(-1)^n I_n(h/D)\,I_n(Jm/D)\sin(n\phi)}{\sum_{n=-\infty}^{\infty}(-1)^n I_n(h/D)\,I_n(Jm/D)\cos(n\phi)}\,g(h)\,dh\,, \tag{18}$$

where $I_n(x)$ is the modified Bessel function of order n. This is main result of our approach. Although these expressions seem to be cumbersome due to the existence of a quotient of infinite series of products of Bessel functions, in practice they are rather simple to deal with because of their excellent properties of convergence. Indeed, less than ten terms of the series are sufficient to compute m and ϕ with very good accuracy. Additionally, it is interesting to notice that h and Jm never appear in the same argument of the Bessel functions, what is appropriate to study the effect of each separately, mainly near the critical point.

4. Discussion and results

The physics of the problem is not difficult to understand. Two basic mechanisms play a relevant role. The ferromagnetic interaction tends to align all the spins since its effect is to decrease the difference of phases between pairs of them. In absence of noise and random field the whole system will be oriented in an arbitrary direction, not fixed a priori. The effect of the random field is to favour or to oppose to this alignment depending on the features of the distribution $g(h)$. It is clear that for positive (negative) semidefinite distributions of random field no transition occurs. There is always a net magnetization (except for $D \to \infty$) in the direction given by the field. However, if the distribution $g(h)$ is even and of zero mean there is a competition between the ferromagnetic interaction and the field which tends to separate the population in two different groups, spatially disordered, each pointing in opposite orientations. Above a critical ratio J_c/D a phase transition occurs and a net magnetization appears spontaneously for $\phi = \pi/2$, i.e perpendicular to the direction fixed by the field.

Let us analyze these situations more carefully. We have considered two different types of distributions: gaussian and uniform. Fig. 1 shows, for an even uniform distribution of zero mean, the variation of the critical point in function of the variance of the distribution. In absence of an external field there is a second order transition in $J = 2D$. As the variance increases the interaction must be larger in order to get a net magnetization. This effect can be observed easily from a linear analysis of the solutions close to the critical point. The expansion of expression (17) up to first order in m, gives

$$J = \frac{2D}{1 - \int [I_2(h/D)/I_0(h/D)]g(h)\,\mathrm{d}h}\,. \tag{19}$$

Moreover, when the field is weak, i.e., if the dispersion of the distribution around zero is sufficiently small as to consider $\sigma^2 > \sigma^n$ for any $n > 2$, then the expansion of the modified Bessel functions in h/D up to second order, gives

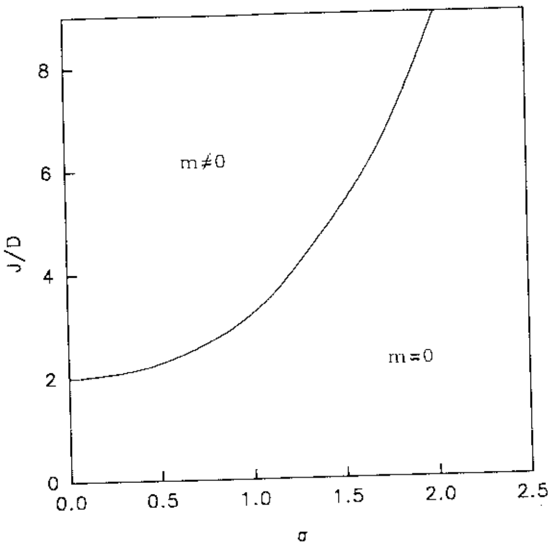

Fig. 1. Phase diagram of a planar *XY* model for an even, zero-mean uniform distribution of random fields. The solid line represents the critical J/D above which a net magnetization appears spontaneously. $\phi = \pi/2$, i.e. perpendicular to the direction given by $g(h)$.

$$J = \frac{16D^3}{8D^2 - \sigma^2}, \tag{20}$$

observing that under these conditions J depends exclusively on the variance. This means that for a given σ^2 the system behaves identically for any type of even distribution $g(h)$.

In order to verify the reliability of the theoretical results given by expressions (17)–(18) we have compared them with simulations. Simulations have been performed by integrating eq. (4) with the Euler method with a time step $\Delta t = 0.002$. We have considered a population of 20000 spins, large enough to neglect finite size effects. Fig. 2 shows the variation of the magnetization m in function of the ratio J/D for both distributions: Gaussian and uniform. Each point in the simulation has been achieved by averaging over one thousand time steps in the stationary state. Notice the excellent agreement between both approaches, what confirms that our results are exact. This is something expected since we have applied a mean-field formalism in a system with infinite range interactions.

For positive (negative) definite distributions there is always a net magnetization. In this case the field favours the alignment of the spins, so that $\phi = 0$ which means that the mean orientation coincides with the direction of the field. In fig. 3 we have plot m versus J/D for a lateral (positive semidefinite) uniform

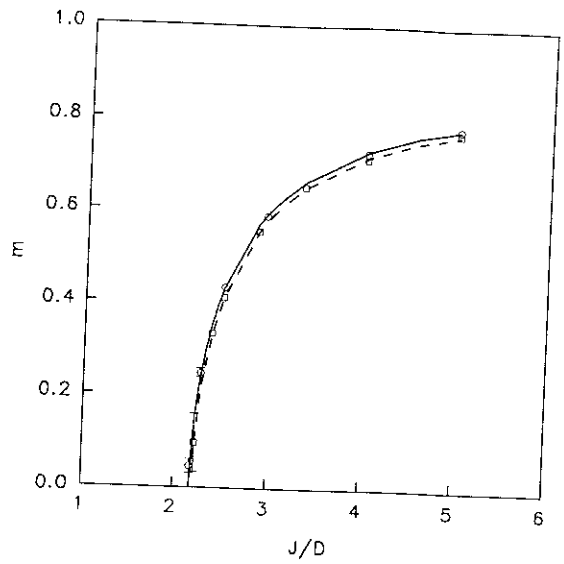

Fig. 2. Magnetization versus J/D for both distributions considered in the paper. Solid and dashed lines represent the theoretical curves given by expressions (17), (18) for a Gaussian and an uniform distribution, respectively. In both cases the variance has been fixed to $\sigma^2 = 0.2$. Circles (Gaussian) and squares (uniform) indicate the results obtained from simulations. Far from the transition point the error bars are of the same order of magnitude of the size of the symbols.

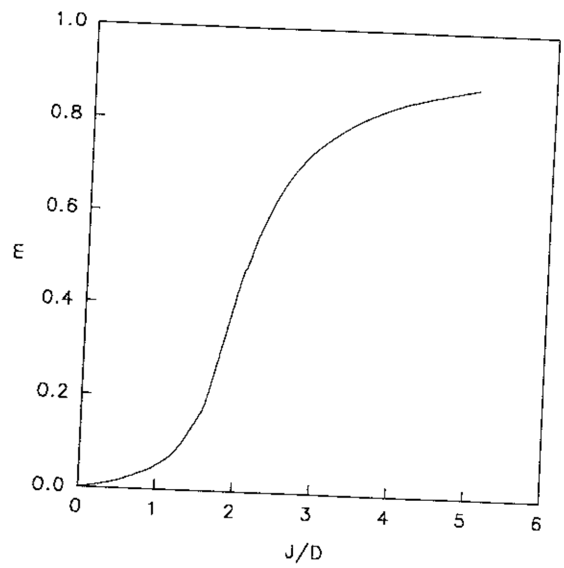

Fig. 3. Magnetization versus J/D for a positive semidefinite uniform distribution $g(h)$, defined in the interval [0,0.1].

distribution. Again, a linear expansion of (17) for small m, h gives the slope of the growth of the magnetization, which turns out to be

$$m = \frac{\bar{h}}{2D - J}, \tag{21}$$

where $\bar{h}$ is the mean value of the considered distribution. Notice also that small m implies $D \gg J$.

In conclusion, we have found the phase diagram of a mean-field planar XY model with relaxational dynamics in presence of quenched disorder induced by a random field. We have applied a Fokker–Planck formalism to analyze the long-time behaviour of the system. We have shown that this technique allows to average over the disorder at the end of the calculation and is appropriate to analyze problems with continuous symmetry.

Acknowledgements

We thank Prof. J. Villain for useful discussions. This work has been partially supported by DGICYT under grant PB92-0863 and EEC Human Capital and Mobility program under contract ERBCHRXCT 930413.

Appendix A

In this appendix we provide a solution of the Fokker–Planck equation (8) in the stationary state. By taking $\partial\rho/\partial t = 0$,

$$\left(\frac{\partial}{\partial\theta}[\omega + Jm\sin(\phi - \theta) + h\sin\theta]\rho\right) - D\frac{\partial^2\rho}{\partial\theta^2} = 0\,. \tag{22}$$

This equation can be easily solved by using the integrating factor

$$\rho = k(\theta)\exp\left[\int L(\theta)\,\mathrm{d}\theta\right]. \tag{23}$$

Defining

$$M(\theta) = \int L(\theta)\,\mathrm{d}\theta = \frac{\omega\theta}{D} + \frac{Jr}{D}\cos(\phi - \theta) - \frac{h}{D}\cos\theta \tag{24}$$

and substituting in (22) we have

$$\rho = A\,\mathrm{e}^{M(\theta)} + B\,\mathrm{e}^{M(\theta)} \int_0^{\theta} \mathrm{e}^{-M(\theta'')}\,\mathrm{d}\theta'' , \tag{25}$$

where A and B are two integration constants which can be determined by imposing normalization and periodicity conditions. Periodicity $\rho(\theta) = \rho(\theta + 2\pi)$ leads to

$$A = \frac{\exp(2\pi\omega/D)}{1 - \exp(2\pi\omega/D)}\,B \int_0^{2\pi} \mathrm{e}^{M(\theta)-M(\theta')}\,\mathrm{d}\theta' . \tag{26}$$

Therefore, we can write ρ as

$$\rho = \frac{B}{1 - \exp(2\pi\omega/D)} \int_{\theta-2\pi}^{\theta} \mathrm{e}^{M(\theta)-M(\theta')}\,\mathrm{d}\theta' . \tag{27}$$

Finally, normalization

$$\int_0^{2\pi} \rho(\theta, \omega, h)\,\mathrm{d}\theta = 1 \tag{28}$$

and the following change of variables: $\theta' - (\theta - 2\pi) = \eta$, leads to the stationary solution for the one-oscillator probability density,

$$\begin{aligned}\rho(\theta) = {} & \exp[(Jm/D)\cos(\phi - \theta) - (h/D)\cos\theta] \\ & \times \int_0^{2\pi} \exp[-(\omega\eta/D) - (Jm/D)\cos(\phi - \theta - \eta) + (h/D)\cos(\theta + \eta)]\,\mathrm{d}\eta \\ & \Big/ \Big(\int_0^{2\pi} \exp[(Jm/D)\cos(\phi - \theta) - (h/D)\cos\theta]\,\mathrm{d}\theta \\ & \times \int_0^{2\pi} \exp[-(\omega\eta/D) - (Jm/D)\cos(\phi - \theta - \eta) + (h/D)\cos(\theta + \eta)]\,\mathrm{d}\eta \Big) .\end{aligned} \tag{29}$$

Appendix B

Appendix B is devoted to the derivation of expressions (17)–(18) from the generating functional $\mathscr{Z}$ defined as

$$\mathscr{Z} = \int_0^{2\pi} \mathrm{d}\theta \; F(\sigma, \psi, \theta) \int_0^{2\pi} H(\theta, \eta) \, \mathrm{d}\eta \,, \tag{30}$$

where

$$F(\sigma, \psi, \theta) = \exp\left(\sigma \cos(\psi - \theta) - \frac{h}{D} \cos \theta\right) \tag{31}$$

and

$$H(\theta, \eta) = \exp\left(-\frac{Jm}{D} \cos(\phi - \theta - \eta) + \frac{h}{D} \cos(\theta + \eta)\right). \tag{32}$$

The first step is to find an algebraic expression for $\mathscr{Z}$. To perform the integrals in η and θ it is convenient to expand the exponential according to

$$\mathrm{e}^{x \cos \beta} = I_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n \cos(n\beta) \, I_n(x) \,, \tag{33}$$

where $I_n(x)$ are the spherical modified Bessel functions of order n. Since the argument of the Bessel functions do not depend on neither θ nor η the calculation of $\mathscr{Z}$ involves exclusively trigonometric integrals which can be readily solved through

$$\int_0^{2\pi} \cos(ax + b) \cos(cx + d) \, \mathrm{d}x = \pi \cos(b - d) \, \delta(a - c) \,, \tag{34}$$

$$\int_0^{2\pi} \sin(ax + b) \sin(cx + d) \, \mathrm{d}x = \pi \cos(b - d) \, \delta(a - c) \,, \tag{35}$$

$$\int_0^{2\pi} \sin(ax + b) \cos(cx + d) \, \mathrm{d}x = \pi \sin(b - d) \, \delta(a - c) \,, \tag{36}$$

where $\delta(x - y)$ is the Kronecker delta. By using the symmetry properties of the modified Bessel functions ($I_n(x) = I_{-n}(x)$) and after some algebra the final expression for the generating functional is

$$\mathscr{Z} = 4\pi \sum_{n=\infty}^{\infty} (-1)^n I_n\left(\frac{h}{D}\right) I_n\left(\frac{kr}{D}\right) \cos(n\phi)$$
$$\times \sum_{m=-\infty}^{\infty} (-1)^m I_m\left(\frac{h}{D}\right) I_m(\sigma) \cos(m\phi) \,. \tag{37}$$

Now, it is straightforward to derive the selfconsistent equations for the order parameters through expressions (15), (16).

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