NOTE ABOUT THE ISOCHRONICITY OF HAMILTONIAN SYSTEMS AND THE CURVATURE OF THE ENERGY

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Abstract. In this note we show the relationship between the period of a isochronous center of a planar Hamiltonian system and the Gauss curvature of the surface $S = (x, y, H(x, y))$ where $H$ is the energy function of the system.

1. Introduction

It is well known that the planar analytic Hamiltonian systems are the planar differential systems of the form

$$\begin{align*}
\dot{x} &= -\frac{\partial H}{\partial y}(x, y) \\
\dot{y} &= \frac{\partial H}{\partial x}(x, y),
\end{align*}$$

where $H$ is an analytic function on $\mathbb{R}^2$. The solutions of these systems are contained in the level curves $\{H(x, y) = h, h \in \mathbb{R}\}$. A point $p$ is called center if it has a neighborhood formed by periodic orbits. The largest neighborhood of $p$ which is entirely covered by periodic orbits is called the period annulus of $p$ and we will denote it by $\mathcal{P}$. The function which associates to any periodic orbit $\gamma$ in $\mathcal{P}$ its period is called the period function. The center is called isochronous center when the period function is a constant $\omega$. It is well known that only nondegenerate centers can be isochronous. And from now on we will assume that $H(0, 0) = 0$ and the system (1.1) has a nondegenerate center at the origin.

Cima, Mañosas and Villadelprat in [1] study the isochronous centers of Hamiltonians systems of the form

$$H(x, y) = A(x) + B(x)y + C(x)y^2$$

with $A, B, C$ analytical functions. They prove that if the center is isochronous of period $\omega$ then

$$\frac{d^2}{dx^2} \left(4AC - B^2\right)(0) = \frac{8\pi^2}{\omega^2}.$$