The transverse structure of Lie flows of codimension 3

By

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1. Introduction

This paper deals with the problem of the realization of a given Lie algebra as transverse algebra to a Lie foliation on a compact manifold.

Lie foliations have been studied by several authors ([E.H.S], [E.N], [F], [H.M], [M], [Ma], etc.). The importance of this study was increased by the fact that they arise naturally in Molino’s classification of Riemannian foliations [M].

To each Lie foliation are associated two Lie algebras, the Lie algebra $\mathfrak{g}$ of the Lie group on which the foliation is modeled and the structural Lie algebra $\mathfrak{k}$. The latter algebra is the Lie algebra of the Lie foliation $\mathcal{F}$ restricted to the closure of any one of its leaves. In particular, it is a subalgebra of $\mathfrak{g}$. We remark that although $\mathfrak{k}$ is canonically associated to $\mathcal{F}$, $\mathfrak{g}$ is not.

Thus two interesting problems are naturally posed: the realization problem and the change problem.

The realization problem is to know which pairs of Lie algebras $(\mathfrak{g}, \mathfrak{k})$, with $\mathfrak{k}$ subalgebra of $\mathfrak{g}$, can arise as transverse and structural Lie algebras, respectively, of a Lie foliation $\mathcal{F}$ on a compact oriented manifold $M$.

This problem is closely related to the following Haefliger’s problem [Ha]: given a Lie subgroup $\Gamma$ of a Lie group $G$, is there a Lie $G$-foliation on a compact manifold $M$ with holonomy group $\Gamma$? E. Ghys [Gh] and G. Méglioz [Mg] also studied this problem and they gave necessary conditions for a pair $(G, \Gamma)$ to be realizable.

Our formulation of the realization problem is a little different: We shall say that the pair $(\mathfrak{g}, q)$ is realizable if there is a compact oriented manifold endowed with a Lie foliation transversely modeled on $\mathfrak{g}$ and with structural Lie algebra of dimension $q$. We also say that $\mathfrak{g}$ is realizable as transverse to a Lie foliation.

This formulation of the realization problem has been considered in [Li], [H], [G, R] and [HLLR] making a very detailed study of Lie flows of codimension $3$ (cf. §8). But a complete classification was not obtained because of the following open question:

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