On the Laplacian Spectrum and Walk-regular Hypergraphs

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Abstract

We use the generalization of the Laplacian matrix to hypergraphs to obtain several spectral-like results on hypergraphs. For instance, we obtain upper bounds on the eccentricity and the excess of any vertex of hypergraphs. We extend to the case of hypergraphs the concepts of walk regularity and spectral regularity, showing that all walk-regular hypergraphs are spectrally-regular. Finally, we obtain an upper bound on the mean distance of walk-regular hypergraphs that involves all the Laplacian spectrum.

Keywords: Laplacian eigenvalues; Walk-regular hypergraphs; Local Spectrum; Excess; Mean distance

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1 Introduction

We begin by stating the used terminology. Throughout this paper, if not otherwise stated \( \mathcal{H} = (V, E) \) denotes a (simple and finite) hypergraph with vertex set \( V = V(\mathcal{H}) \), \(|V| = n\), and edge set \( E = E(\mathcal{H}), |E| = m \). The degree \( \delta(v) \) of the vertex \( v \) is defined as the cardinality of the edge set containing \( v \). A hypergraph in which all vertices have the same degree, \( \delta \), is called \( \delta\text{-regular} \), and if all edges have the same cardinality \( r \), it is called \( r\text{-uniform} \).

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The distance \( \partial(u,v) \) between two vertices \( u \) and \( v \) is the minimum of the lengths of paths between \( u \) and \( v \). The eccentricity of a vertex \( v \) is defined as
\[
\epsilon = \max_{u \in V(\mathcal{H})} \partial(u,v)
\]
and the diameter \( D(\mathcal{H}) \) of a hypergraph \( \mathcal{H} \) is defined as
\[
D(\mathcal{H}) = \max_{v \in V(\mathcal{H})} \epsilon(v).
\]
The reader is referred to [2] for more details on hypergraphs.

The plan of the paper is the following: In Section 2 we recall the definition of the Laplacian matrix \( \mathbf{L} \) of a hypergraph and we emphasize some of its main properties, giving the main tool of the paper, that is, if \( \partial(v_i,v_j) > k \) then, for any real polynomial \( P \) of degree \( \leq k \), we have \( (P(\mathbf{L}))_{ij} = 0 \). In Section 4 we extend to the case of hypergraphs the concepts of walk regularity and spectral regularity, showing that all walk-regular hypergraphs are spectrally-regular. In Section 4 we define the local Laplacian polynomials that we will use as tool in Section 5. Finally, Section 5 is devoted to obtain upper bounds of parameters related with the concept of distance in hypergraphs. More specifically, upper bounds on the eccentricity and the excess of any vertex of hypergraphs are given. Moreover, we obtain an upper bound on the mean distance of walk-regular hypergraphs that involve all the Laplacian spectrum.

## 2 Laplacian Matrix

We denote by \( \mathbf{A} = \mathbf{A}(\mathcal{H}) \) the adjacency matrix of the hypergraph \( \mathcal{H} \). Given two distinct vertices \( v_i, v_j \in V(\mathcal{H}) \) the entry \( a_{ij} \) of \( \mathbf{A} \) is the number of edges in \( \mathcal{H} \) containing both \( v_i \) and \( v_j \); the diagonal entries of \( \mathbf{A} \) are zero. We define the Laplacian degree of a vertex \( v_i \in V(\mathcal{H}) \) as
\[
\delta_\ell(v_i) = \sum_{j=1}^{n} a_{ij}.
\]
We say that the hypergraph \( \mathcal{H} \) is Laplacian regular of degree \( \delta_\ell \) if any vertex \( v \in V(\mathcal{H}) \) has Laplacian degree \( \delta_\ell(v) = \delta_\ell \). A simple count shows that the Laplacian degree of an \((n,r,\delta)\)-design\(^1\) satisfies
\[
\delta_\ell = (r - 1)\delta = \frac{mr(r - 1)}{n}.
\]
\(^1\)An \( r \)-uniform, \( \delta \)-regular hypergraph, of order \( n \), is called \((n,r,\delta)\)-design
Moreover, if $\mathcal{H}$ is a graph then $\delta_{\ell}(v_i) = \delta(v_i)$.

The Laplacian matrix of a hypergraph $\mathcal{H}$, denoted by $L = L(\mathcal{H})$, is defined (see [16]) as $L = O - A$ where $O = \text{diag}(\delta_{\ell}(v_1), \delta_{\ell}(v_2), \ldots, \delta_{\ell}(v_n))$. We recall that the matrix $L$ is symmetric and positive semidefinite, the smallest eigenvalue of $L$ is $\mu = 0$ and a corresponding eigenvector is $j = (1, 1, \ldots, 1)$. Moreover, the multiplicity of $\mu = 0$ is equal to the number of components of $\mathcal{H}$. In this paper we only consider the case of connected hypergraphs.

The eigenvalues of $L$ are denoted by $\mu_0 < \mu_1 < \cdots < \mu_b$ and their multiplicities are denoted by $m_0 = m(\mu_0) = 1, m_1 = m(\mu_1), \ldots, m_b = m(\mu_b)$. Thus, the Laplacian spectrum of $\mathcal{H}$ is denoted by

$$\text{Spec}(\mathcal{H}) = \{\mu_0^1, \mu_1^{m_1}, \ldots, \mu_b^{m_b}\}.$$  

The Laplacian spectrum of $\mathcal{H}$ and the Laplacian degrees of its vertices are related by the following equality

$$\sum_{l=1}^{b} m_l \mu_l = \text{Tr}(L) = \sum_{i=1}^{n} \delta_{\ell}(v_i). \quad (1)$$

We denote by $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ the adjacency eigenvalues of a Laplacian regular hypergraph $\mathcal{H}$ of degree $\delta_{\ell}$. Then, since $L = \delta_{\ell}I - A$, the eigenvalues of both matrices, $A$ and $L$, are related by

$$\mu_l = \delta_{\ell} - \lambda_l, \quad l = 0, \ldots, b = d. \quad (2)$$

Notice also that $\delta_{\ell}$ is the trivial eigenvalue of $A$ with $j$ as eigenvector. Hence, in this case, the matrices $A$ and $L$ lead to equivalent spectral-like results.

We identify the Laplacian matrix $L$ with an endomorphism of the “vertex-space” of $\mathcal{H}$, $l^2(V(\mathcal{H}))$ which, for any given indexing of the vertices, is isomorphic to $\mathbb{R}^n$. Thus, for any vertex $v_i \in V(\mathcal{H})$, $e_i$ will denote the corresponding unit vector of the canonical base of $\mathbb{R}^n$.

### 2.1 Local Spectrum

In [10] Fiol, Garriga and Yebra defined and studied the properties of the local spectrum of a graph. Here we extend this concept to the case of the Laplacian matrix of a hypergraph.
For a given vertex \( v_i \) we can consider the spectral decomposition of the corresponding unit vector \( e_i \)

\[
e_i = \sum_{l=0}^{b_i} z_{il}, \quad \text{where} \quad z_{il} \in \text{Ker}(L(H) - \mu_l I).
\]

(3)

The \( v_i \)-local multiplicity of the Laplacian eigenvalue \( \mu_l \) is defined as

\[
m_i(\mu_l) = \|z_{il}\|^2.
\]

Thus, the \( v_i \)-local multiplicity of \( \mu_0 = 0 \) is \( m_i(0) = \frac{1}{n} \). The \( v_i \)-local eigenvalues of \( L \) are denoted by \( 0 < \xi_1 < \cdots < \xi_{b_i} \) and they are defined as the Laplacian eigenvalues with nonnull \( v_i \)-local multiplicities. The \( v_i \)-local spectrum of \( L \) is defined as

\[
\text{Spec}_i(L) = \left\{ 0^n, \xi_1^{m_i(\xi_1)}, \ldots, \xi_{b_i}^{m_i(\xi_{b_i})} \right\}.
\]

When we “see” the hypergraph from a given vertex, its local spectrum plays a similar role as the global spectrum, thus justifying the terminology used. We collect here some of its main properties, referring the reader to [10] for an analogous and detailed study. In general, for any real polynomial, \( P \), using (3), we have

\[
(P(L))_{ii} = \langle P(L)e_i, e_i \rangle
\]

\[
= \left\langle P(L) \sum_{l=0}^{b_i} z_{il}, \sum_{l=0}^{b_i} z_{il} \right\rangle
\]

\[
= \left\langle \sum_{l=0}^{b_i} P(\mu_l)z_{il}, \sum_{l=0}^{b_i} z_{il} \right\rangle
\]

\[
= \sum_{l=0}^{b_i} P(\mu_l)\|z_{il}\|^2
\]

\[
= \sum_{l=0}^{b_i} P(\xi_l)m_i(\xi_l).
\]

Thus, we conclude

\[
(P(L))_{ii} = \sum_{l=0}^{b_i} P(\xi_l)m_i(\xi_l).
\]

(4)
In particular, if \( P(x) \equiv 1 \), (4) leads to
\[
\sum_{l=0}^{b_i} m_i(\xi_l) = 1. \tag{5}
\]
Moreover, if \( P(x) = x \), (4) leads to
\[
\sum_{l=0}^{b_i} \xi_l m_i(\xi_l) = \delta_i(v_i). \tag{6}
\]

### 2.2 Laplacian Matrix and Distances

We will show that, as in graphs, there is a close connection between the entries of the powers of the adjacency (Laplacian) matrix of a hypergraph and the number of walks between two vertices.

**Lemma 1.** Let \( v_i \) and \( v_j \) be vertices of a hypergraph \( \mathcal{H} \). Let \( A \) be the adjacency matrix of \( \mathcal{H} \). Then, the number of walks of length \( k \) in \( \mathcal{H} \), from \( v_i \) to \( v_j \), is the entry in position \((i, j)\) of the matrix \( A^k \).

**Proof.** The result is true for \( k = 0 \) and \( k = 1 \), since \( A^0 = I \) and \( A^1 = A(\mathcal{H}) \). Suppose that the result is true for \( k = N \). Let \( v_h \) and \( v_j \) be adjacent vertices, then the number of walks of length \( N + 1 \) from \( v_i \) to \( v_j \) containing \( v_h \) is \((A^N)_{ih} \cdot a_{hj}\). Thus, the number of walks of length \( N + 1 \) from \( v_i \) to \( v_j \) is
\[
\sum_{v_h \sim v_j} (A^N)_{ih} \cdot a_{hj} = \sum_{h=1}^{n} (A^N)_{ih} \cdot a_{hj} = (A^{N+1})_{ij}. 
\]
The general result follows by induction. \( \square \)

Our main tool in this paper is the following basic theorem.

**Theorem 2.** Let \( v_i \) and \( v_j \) be vertices of a hypergraph \( \mathcal{H} \). Let \( P \) be a real polynomial of degree \( k \). Then,
\[
\partial(v_i, v_j) > k \implies (P(L))_{ij} = 0.
\]

**Proof.** By definition of Laplacian matrix we have
\[
L^k = (O - A)^k. \tag{7}
\]
If \( \partial(v_i, v_j) > k \), Lemma 1 and the expansion of (7) lead to \((L^k)_{ij} = 0\) (since \( O \) is a diagonal matrix). Thus, the result immediately follows. \( \square \)
3 Walk-regular Hypergraphs

A hypergraph is walk-regular if for every $k$ the number of walks of length $k$ with both endpoints at $v$ does not depend on the vertex $v$. In other words, any power $A^k$ has its diagonal entries all equal to $Tr(A^k)/n$. Obviously, every walk-regular hypergraph is Laplacian regular. The class of walk-regular graphs contains the class of vertex transitive graphs and the class of distance-regular graphs.

In the case of the Laplacian matrix we have the following fact.

Lemma 3. Any power $L^k$ of the Laplacian matrix of a walk-regular hypergraph has its diagonal entries all equal to $Tr(L^k)/n$.

Proof. By definition of Laplacian matrix of a Laplacian regular hypergraph we have

$$L^k = (\delta I - A)^k = \sum_{r=0}^{k} \binom{k}{r} \delta^{k-r}(-A)^r.$$

Then

$$(L^k)_{ii} = \sum_{r=0}^{k} \binom{k}{r} \delta^{k-r}(-1)^r \frac{Tr(A^r)}{n}.$$ 

The result immediately follows.

Like in the case of graphs (see Fiol and Garriga in [12] for the case of graphs), we say that a hypergraph $H$ is spectrally-regular when all its vertices have the same local spectrum, that is, when for any pair of vertices $v_i, v_j$, we have $m_i(\mu_l) = m_j(\mu_l)$ for any $l = 0, 1, \ldots, b$. In particular, $b_i = b$, for any $i$.

Theorem 4. Let $H$ be a walk-regular hypergraph. Then, $H$ is spectrally-regular and the global and local multiplicities of its Laplacian eigenvalues are related by

$$m_i(\mu_l) = \frac{m(\mu_l)}{n}, \quad l = 0, 1, \ldots, b, \quad i = 1, \ldots, n.$$ (8)

Proof. In the case of $\mu_0 = 0$ we have $m_i = \frac{1}{n} = \frac{m_0}{n}$ $(i = 1, \ldots, n)$. In the remaining cases, by Lemma 3 we have

$$Tr(L^k) = n(L^k)_{ii},$$

6
and using (4), when $P(x) = x^k$, $k = 1, \ldots, b$, we obtain

$$(L^k)_{ii} = \sum_{l=0}^{b} \xi_l^k m_i(\xi_l) = \sum_{l=0}^{b} \mu_l^k m_i(\mu_l).$$

So, we have

$$\text{Tr} \left( L^k \right) = n \sum_{l=1}^{b} \mu_l^k m_i(\mu_l).$$

On the other hand,

$$\text{Tr} \left( L^k \right) = b \sum_{l=1}^{b} \mu_l^k m_i(\mu_l).$$

Therefore,

$$\sum_{l=1}^{b} \mu_l^k \left( nm_i(\mu_l) - m(\mu_l) \right) = 0, \quad k = 1, \ldots, b.$$ 

The above linear equations system has nonnull determinant. Hence the result follows.

In the case of the adjacency spectra of walk-regular graphs, the analogous result of (8) was obtained by Delorme and Tillich in [7] and by Fiol, Garriga and Yebra in [11].

The above theorem leads to a procedure to compute the independent term of the Laplacian polynomials.

4 Laplacian Polynomials

In this section, by using the $v_i$-local spectrum of $L$, we define the $v_i$-local $k$-Laplacian polynomials that we will use as tool in the following sections. The study of these polynomials is completely analogous to the study of the local adjacency polynomials defined and studied by Fiol and Garriga in [12], therefor, we collect here some of its main properties, referring the reader to [12] for a more detailed study, detailing here only the case of walk-regular hypergraphs.

Let $\xi_0 = 0 < \xi_1 < \cdots < \xi_b$ be the $v_i$-local eigenvalues of $L$. For each $k = 0, \ldots, b_i$, the mapping $\| \|_i: \mathbb{R}_k[x] \rightarrow \mathbb{R}$ defined by $\| P \|_i = \| P(L)e_i \|$ is a norm of the space $\mathbb{R}_k[x]$. In this normed space, we consider the closed
unit ball $B_k = \{ P \in \mathbb{R}_k[x] : \|P\|_i \leq 1 \}$. On this compact set, the linear continuous function $P \mapsto P(0)$ attains its maximum at a point $q_k^0$, which we call $v_i$-local $k$-Laplacian polynomial. Notice that, such a point must be on the border of $B_k$; that is, $\|q_k^0\|_i = 1$.

Between the main properties of the $v_i$-local $k$-Laplacian polynomials we emphasize the following.

- Each $v_i$-local $k$-Laplacian polynomial has degree $k$.
- $1 = q_0^i(0) < q_1^i(0) < \cdots < q_{b_i}^i(0) = \sqrt{n}$.

As we are going to see in the following sections, in practice, we only need the independent term $q_k^i(0)$ of $q_k^i$.

Now we emphasize the case of walk-regular hypergraphs. By definition of $q_k^i$ we have that if $q_k^i(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_k x^k$ then

$$1 = \|q_k^i\|_i^2 = \|q_k^i(L)e_i\|^2 = \frac{1}{n} \alpha_0^2 + \sum_{l=1}^{b_i} m_i(\xi_l)(\alpha_0 + \alpha_1 \xi_l + \cdots + \alpha_k \xi_l^k)^2.$$

However, in the case of walk-regular hypergraphs, by Theorem 4, we have

$$1 = \alpha_0^2 \frac{1}{n} + \sum_{l=1}^{b_i} \frac{m_l}{n} (\alpha_0 + \alpha_1 \mu_l + \cdots + \alpha_k \mu_l^k)^2.$$

Therefore, in this case, the polynomials $q_k^i$ do not depend on $v_i$. Hence we call these polynomials $k$-Laplacian polynomials and they will be denoted as $q_k$.

Consequently, the independent term of $q_k$ can be calculated by the following constrained optimization problem:

maximize $\alpha_0$, subject to

$$\alpha_0^2 + \sum_{l=1}^{b_i} m_l(\alpha_0 + \alpha_1 \mu_l + \cdots + \alpha_k \mu_l^k)^2 = n.$$

The objective to use the $v_i$-local $k$-Laplacian polynomials, and not others, is the following: If we compute the bound for the excess of the vertex $v_i$ by the procedure used in the proof of Theorem 8, using a generic polynomial $P$ of degree $k$, we obtain

$$e_k(v_i) \leq \left[ n - \frac{(P(0))^2}{\|P(L)e_i\|^2} \right].$$

Hence, by using the $v_i$-local $k$-Laplacian polynomials we obtain the optimum result.
5 Bounding metric parameters

5.1 Bounding the Eccentricity

Recently, several results bounding above the eccentricity of a vertex or the
diameter from the eigenvalues of either the adjacency matrix or the Laplacian
matrix of the graph have appeared in the literature. In the case of the
eccentricity we cite, Fiol and Garriga [11] and Yebra and the author in [17].
In the case of the diameter we cite, for instance, Chung [4], Quenell [14],
Delorme and Solé [6], Delorme and Tillich [7] Van Dam and Haemers [5],
Alon and Milman [1], Mohar [13], Fiol, Garriga and Yebra [8, 9], Fiol and
Garriga [11]. In the case of the diameter of hypergraphs we cite the author
in [16].

We begin by giving the relation between the eccentricity of a vertex \( v_i \)
and the number \( b_i + 1 \) of \( v_i \)-local eigenvalues of \( L \).

**Theorem 5.** Let \( b_i + 1 \) be the number of different \( v_i \)-local eigenvalues of \( L \). Then,

\[
\epsilon(v_i) \leq b_i.
\]

**Proof.** Let \( e_i \) be the canonical vector of \( \mathbb{R}^n \) associated to the vertex \( v_i \). By
using the following decomposition

\[
e_i = \sum_{l=0}^{b_i} z_{il}, \quad \text{where} \quad z_{il} \in \text{Ker}(L - \xi_l I),
\]

and the polynomial

\[
P(x) = \prod_{l=1}^{b_i} (x - \xi_l)
\]

we have

\[
(P(L))_{ij} = \langle P(L)e_i, e_j \rangle = \sum_{l=0}^{b_i} P(\xi_l) \langle z_{il}, z_{jl} \rangle = \frac{P(0)}{n} = \frac{(-1)^{b_i}}{n} \prod_{l=1}^{b_i} \xi_l \neq 0
\]

Then, by the converse of Theorem 2 the result follows. \( \square \)

In [16] we showed that if the Laplacian matrix of a hypergraph \( H \) has
\( b + 1 \) different eigenvalues then \( D(H) \leq b \). This fact is well-known in the
particular case of graphs.
**Theorem 6.** Let $v_i$ be a vertex of a hypergraph of order $n$, and let $q_k^i$ be its $v_i$-local $k$-Laplacian polynomial. Then,

$$q_k^i(0) > \sqrt{n-1} \Rightarrow \epsilon(v_i) \leq k.$$ 

**Proof.** Let $e_i$ and $e_j$ be the canonical vectors of $\mathbb{R}^n$ associated to the vertices $v_i$ and $v_j$ respectively. By Theorem 2 we have

$$\partial(v_i, v_j) > k \Rightarrow (q_k^i(L))_{ij} = 0.$$ 

Using the following decomposition

$$e_i = \frac{j}{n} + u, \quad \text{and} \quad e_j = \frac{j}{n} + w, \quad \text{where} \quad u, w \in j^\perp \quad (10)$$

we obtain

$$0 = (q_k^i(L))_{ij} = \langle q_k^i(L)e_i, e_j \rangle = q_k^i(0)\frac{1}{n} + \langle q_k^i(L)u, w \rangle.$$ 

Hence,

$$q_k^i(0) = -\langle q_k^i(L)u, w \rangle.$$ 

Then, by the Cauchy-Schwarz inequality we have

$$\frac{q_k^i(0)}{n} \leq \|q_k^i(L)u\| \|w\|. \quad (11)$$

Moreover, the decomposition (10) leads to

$$1 = \|e_j\|^2 = \frac{1}{n} + \|w\|^2 \Rightarrow \|w\| = \sqrt{\frac{n-1}{n}}$$

and

$$1 = \|q_k^i(L)e_i\|^2 = \frac{(q_k^i(0))^2}{n} + \|q_k^i(L)u\|^2 \Rightarrow \|q_k^i(L)u\| = \sqrt{\frac{n-(q_k^i(0))^2}{n}}.$$ 

So, by (11), we obtain

$$q_k^i(0) \leq \sqrt{n-1}.$$ 

Therefore,

$$\partial(v_i, v_j) > k \Rightarrow q_k^i(0) \leq \sqrt{n-1}. \quad (12)$$ 

The converse of (12) leads to the result. \qed
As we showed in Section 4, the \(v_i\)-local \(k\)-Laplacian polynomial of a walk-regular hypergraph does not depend on the vertex \(v_i\). Then, the above theorem automatically gives a bound on the diameter.

**Corollary 7.** Let \(\mathcal{H}\) be a walk-regular hypergraph and let \(q_k\) be its \(k\)-Laplacian polynomial. Then,

\[q_k(0) > \sqrt{n-1} \Rightarrow D(\mathcal{H}) \leq k.\]

### 5.2 Bounding the Excess

We define for any \(k = 0, 1, \ldots, D(\mathcal{H})\), the \(k\)-excess of a vertex \(v_i \in V(\mathcal{H})\), denoted by \(e_k(v_i)\), as the number of vertices which are at distance greater than \(k\) from \(v_i\). That is,

\[e_k(v_i) = |\{v_j \in V : \partial(v_i, v_j) > k\}|.\]

Then, trivially, \(e_0(v_i) = n - 1\), \(e_{D(\mathcal{H})}(v_i) = e_{\epsilon(v_i)}(v_i) = 0\) and \(e_k(v_i) = 0\) if and only if \(\epsilon(v_i) \leq k\), where \(\epsilon(v_i)\) denote the eccentricity of \(v_i\). The name “excess” is borrowed from Biggs [3], in which he gives a lower bound, in terms of the adjacency eigenvalues of a graph, for the excess \(e_r(v_i)\) of any vertex \(v_i\) in a regular graph with odd girth. The excess of a vertex of a graph was studied by Fiol and Garriga [12] using the adjacency eigenvalues of a graph and Yebra and the author [17] using the Laplacian eigenvalues.

The \(k\)-excess of \(\mathcal{H}\) denoted by \(e_k\), is defined as

\[e_k = \max_{v_i \in V(\mathcal{H})} \{e_k(v_i)\}.\]

In this section we obtain an upper bound for the \(k\)-excess \(e_k(v_i)\) of any vertex of \(\mathcal{H}\) and, in particular, for the \(k\)-excess \(e_k\) of \(\mathcal{H}\), when the hypergraph \(\mathcal{H}\) is walk-regular.

**Theorem 8.** Let \(v_i\) be a vertex of a hypergraph of order \(n\), and let \(q_k^i\) be its \(v_i\)-local \(k\)-Laplacian polynomial. Then,

\[e_k(v_i) \leq \left\lfloor n - (q_k^i(0))^2 \right\rfloor.\]

**Proof.** Let \(W\) be a vertex set such that \(\partial(v_i, v_j) > k\) for all \(v_j \in W\). In other words \(\partial(v_i, W) > k\). Then, Theorem 2 leads to

\[\langle q_k^i(L)e_i, w \rangle = 0, \quad \text{where} \quad w = \sum_{v_j \in W} e_j.\]
Suppose that $|W| = e_k(v_i)$. By the following decomposition
\[
e_i = \frac{j}{n} + u, \quad \text{and} \quad w = \frac{e_k(v_i)}{n} j + z, \quad \text{where} \quad u, z \in j^\perp \tag{13}
\]
we obtain
\[0 = \langle q_k(L)e_i, w \rangle = \frac{q_k(0)e_k(v_i)}{n} + \langle q_k(L)u, z \rangle.
\]Then, by the Cauchy-Schwarz inequality we have
\[\frac{q_k(0)e_k(v_i)}{n} \leq \|q_k(L)u\|\|z\|. \tag{14}\]
Moreover, by the decomposition (13) we obtain
\[
\|q_k(L)u\| = \sqrt{\frac{n - (q_k(0))^2}{n}} \quad \text{and} \quad \|z\| = \sqrt{\frac{ne_k(v_i) - (e_k(v_i))^2}{n}}. \tag{15}\]
Finally, (14) and (15) lead to the result.

\[\square\]

\textbf{Corollary 9.} Let $\mathcal{H}$ be a walk-regular hypergraph and let $q_k$ be its $k$-Laplacian polynomial. Then,
\[e_k \leq \lfloor n - (q_k(0))^2 \rfloor.\]

\subsection*{5.3 Bounding the Mean distance}

The distance of a vertex $v$ in a connected hypergraph $\mathcal{H}$ is defined by
\[S(v) = \sum_{u \in V(\mathcal{H})} \partial(u, v).
\]
Clearly,
\[S(v) = \sum_{k=1}^{D(\mathcal{H})} k(e_{k-1}(v) - e_k(v)).
\]
Moreover, by a simple calculation we have
\[S(v) = \sum_{k=0}^{D(\mathcal{H})-1} e_k(v). \tag{16}\]
Hence we can obtain bounds for the distance of a vertex \( v \) from bounds on the excess of \( v \).

The mean distance \( D_m(\mathcal{H}) \) of a hypergraph \( \mathcal{H} \) is the average of all distances between distinct vertices of \( \mathcal{H} \). That is,

\[
D_m(\mathcal{H}) = \frac{1}{n(n-1)} \sum_{v \in V(\mathcal{H})} S(v).
\]

Then, by (16) we have

\[
D_m(\mathcal{H}) = \frac{1}{n(n-1)} \sum_{v \in V(\mathcal{H})} \sum_{k=0}^{D(\mathcal{H})-1} e_k(v)
\]

Indeed, it follows from (17) that bounds on the \( k \)-excess of \( \mathcal{H} \) lead to bounds on its mean distance. One of them is the following.

**Theorem 10.** Let \( \mathcal{H} \) be a walk-regular hypergraph and let \( q_k \) be its \( k \)-Laplacian polynomial. Then,

\[
D_m(\mathcal{H}) \leq \frac{1}{n-1} \sum_{k=0}^{D(\mathcal{H})-1} \lfloor n - (q_k(0))^2 \rfloor.
\]

**Proof.** By (17) and Corollary 9 we have

\[
D_m(\mathcal{H}) \leq \frac{1}{n-1} \sum_{k=0}^{D(\mathcal{H})-1} \lfloor n - (q_k(0))^2 \rfloor.
\]

Using Corollary 7 we conclude the proof. \( \square \)

The mean distance of a graph was studied by Mohar [13], and Yebra and the author [15]. In the case of hypergraph we showed (see [16]) that

\[
D_m(\mathcal{H}) \leq \frac{1}{n-1} \sum_{k=0}^{D(\mathcal{H})-1} \left\lfloor \frac{n(n-1)}{Q_k^2(0)} + n - 1 \right\rfloor,
\]

where \( Q_k \) denotes the \( k \)-alternating polynomial defined over the Laplacian eigenvalues of a not necessarily regular hypergraph \( \mathcal{H} \). There are hypergraphs in which the bound (18) is attained (the bound is tight, see [16]). However, as we will see later, there are walk-regular hypergraphs in which Theorem 10 leads to better results.
5.4 An example

Let $H$ be the $(12,3,3)$-design whose blocks are

- $b_1 = \{1, 2, 3\}$
- $b_2 = \{2, 3, 4\}$
- $b_3 = \{3, 4, 5\}$
- $b_4 = \{4, 5, 6\}$
- $b_5 = \{5, 6, 7\}$
- $b_6 = \{6, 7, 8\}$
- $b_7 = \{7, 8, 9\}$
- $b_8 = \{8, 9, 10\}$
- $b_9 = \{9, 10, 11\}$
- $b_{10} = \{10, 11, 12\}$
- $b_{11} = \{11, 12, 1\}$
- $b_{12} = \{12, 1, 2\}$.

The hypergraph $H$ is walk-regular, that is,

\[ \delta_\ell = \frac{Tr(L)}{12} = 6, \quad \frac{Tr(L^2)}{12} = 46, \quad \frac{Tr(L^3)}{12} = 372, \quad \frac{Tr(L^4)}{12} = 3078, \ldots \]

The Laplacian spectrum of $H$ is

\[ \text{Spec}(H) = \left\{ 0^1, (5 - 2\sqrt{3})^2, 5^2, 8^3, (5 + 2\sqrt{3})^2, 9^2 \right\}. \]

Thus, by solving the constrained optimization problem on page 8, we calculated the independent terms of $q_k$, $k = 0, \ldots, 3$:

\[ q_0(0) = 1, \quad q_1(0) = \sqrt{\frac{23}{5}}, \quad q_2(0) = \frac{3\sqrt{8989}}{101}, \quad q_3(0) = \frac{2\sqrt{137197647744967}}{7040809}. \]

As $q_3(0) > \sqrt{11}$, Corollary 7 gives $D(H) \leq 3$ and the bound is attained. Similarly, by Corollary 9 we get the following bounds to the excess: $e_0 \leq 11$, $e_1 \leq 7$ and $e_3 \leq 0$; the bounds are attained. In the case $k = 2$ we have $e_2 = 3$ while the Corollary 9 only gives $e_2 \leq 4$. Moreover, the mean distance of $H$ is $D_m(H) = 1.909$, by Theorem 10, $D_m(H) \leq 2$ while by (18) we only get $D_m(H) \leq 3.45$.

References


