CARTESIAN APPROACH FOR NONHOLONOMIC SYSTEMS

Rafael Ramírez[1] and Natalia Sadovskaia[2]

[2] Departamento de Ingenería Informática
Universitat Rovira i Virgili de Tarragona
rafaelorlando.ramirez@ urv.cat
[2] Departamento de Matemática Aplicada II
Universitat Politècnica de Catalunya
natalia.sadovskaia@upc.edu

Abstract

Abstract.
In the history of mechanics, there have been two points of view for studying mechanical systems: The Newtonian and the Cartesian. The Cartesian point of view affirms (by using the modern mathematical language) that it is possible to solve the dynamics problem inside the configuration space. In this paper we develop the Cartesian approach for mechanical systems with constraints which are linear with respect to velocity. The obtained results are illustrated in several examples. In particular, we analyze the behavior of the mechanical system with three degrees of freedom and one non-integrable constraint such as the Chapliguin-Carateodory sleigh, the heavy rigid body in the Suslov case, the nonholonomically constrained particle in R^3, and the mechanical system with N degrees of freedom and $N-1$ integrable constraints (we propose a complete solution of the inverse problem in dynamics).

Key words: Cartesian approach, Newtonian approach, constraint, differential equation, Lagrangian systems, nonholonomic systems, optimal control problem, Lie algebra, Inverse problem in Dynamics.

Mathematics Subject Classifications (2000) 00A71, 34A34, 37J60.

1. Introduction

In the history of mechanics, there have been two points of view for studying mechanical systems: The Newtonian and the Cartesian. In "Philosophiae Naturalis Principia Mathematica" (1687), Newton considers that movements of celestial bodies can be described by differential equations of the second order. To determine their trajectory, it is necessary to give the initial position and velocity.

Descartes proposed that the behavior of the celestial bodies be studied from another point of view. These ideas were stated in "Principia Philosophiae" (1644) and in "Discours de la métode" (1637). According to Descarte the understanding of cosmology starts from acceptance of the initial chaos, whose moving elements are ordered according to certain fixed laws and form the Cosmo. He consider that the Universe is filled with a tenuous fluid matter (ether), which is constantly in a vortex motion. This motion moves the largest particle of matter of the vortex axis, and they subsequently form planets. Then, according to what Descartes wrote in his "Treatise on Light", "the material of the Heaven must be rotate the planets not only about the Sun but also about their own centers...and this will hence form several small Heavens rotating in the same direction as the great Heaven."

Newton gave a simpler, but stronger, argument against Descarte's theory. If the Descarte's ideas is correct, bodies are carried by the ether, and the equations of motion are consequently of first order: the velocity of a particle depend only on its position. However, Newton noted that some of the observed comets move in a direction opposite to that of all the planets [Kozlov].

In the modern scientific literature the study of the Descarte ideas we can find in the monographic of V.V. Kozlov in which the author said "In the present book, one more attempt is made to rehabilitate Descarte's vortex theory..." . In this books, Kozlov affirms "solving dynamics problem is possible inside the configuration space".

As we observe , the equation of motion in the Descartes theory must be of the first order

$$\dot{\mathbf{x}} = \mathbf{v}(x) \tag{1.1}$$

Hence, to determine the trajectory from Descartes's point of view it is necessary to give only the initial position. Descarte gave no principles for constructing the field $\mathbf{v}$ for different mechanical systems.

A main achievement of Newton was perceiving that the dynamics of real systems are described by second-order differential equations. To deduce the equations of motion to the investigation of a dynamics systems (i.e., to first order equation), it is necessary to double the dimension of the position space and to introduce the auxiliary phase space. However, we are interested not in the phase trajectories themselves but in their projection on the configuration space.

Definition

The vector field (1.1) we shall call the Cartesian vector field.

The aim of the present paper is to develop the Descarte ideas for mechanical systems with constraints which are linear with respect to the velocity.

2. CONSTRUCTION THE CARTESIAN VECTOR FIELD FOR NONHOLONOMIC SYSTEM

Firstly we shall introduce the following notation and concept.

Let $\mathcal{Q}$ be a smooth manifold of the dimension N with local coordinates $x = (x^1, ..., x^N)$ and equipped by the Riemann metric $G = (G_{kj}(x))$.

By $\xi(\mathcal{Q})$, $\Lambda(\mathcal{Q})$, ∇ we denote respectively the Lie algebra of vector fields on $\mathcal{Q}$ and the algebra of the 1-form on $\mathcal{Q}$, and the connection:

$$\nabla : \xi(\mathcal{Q}) \times \xi(\mathcal{Q}) \longmapsto \xi(\mathcal{Q})$$
$$(u, v) \longmapsto \nabla_u v$$

which is R lineal with respect to v and C^∞ lineal with respect to u and is compatible with metric G, i.e., $\nabla_u G(v, w) = 0, \quad \forall u, v, w \in \xi(\mathcal{Q})$.

Let $\mathbf{v} \in \xi(\mathcal{Q})$ be a vector field:

$$\mathbf{v} = det \begin{pmatrix} \Omega_1(\partial_1) & \Omega_1(\partial_2) & \dots & \Omega_1(\partial_N) & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_M(\partial_1) & \Omega_M(\partial_2) & \dots & \Omega_M(\partial_N) & 0 \\ \Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \dots & \Omega_{M+1}(\partial_N) & \lambda_{M+1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_N(\partial_1) & \Omega_N(\partial_2) & \dots & \Omega_N(\partial_N) & \lambda_N \\ \partial_1 & \partial_2 & \dots & \partial_N & 0 \end{pmatrix},$$

where $\partial_k = \frac{\partial}{\partial x^k}$, we shall consider that $\Omega_1, \Omega_2, ...\Omega_M, \quad M \leq N-1$ are given 1-forms, and $\Omega_{M+1}, \Omega_{M+2}, ...\Omega_N$, are arbitrary 1-forms on $\mathcal{Q}$. Furthermore, we assume that they are pointwise independent i.e.

$$\Upsilon \equiv \Omega_1 \wedge \Omega_2 ... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N) \neq 0,$$

The functions $\lambda_j, \quad j = M+1, ..., N$ are arbitrary functions on $\mathcal{Q}$

The vector field $\mathbf{v}$ has the following properties

1.

$$\begin{cases} \Omega_j(\mathbf{v}) = 0, \quad j = 1, 2, .., M \\ \Omega_j(\mathbf{v}) = -\Upsilon \lambda_j, \quad j = M+1, .., N. \end{cases}$$

2.The vector $\mathbf{v}(x) = \left(v^1(x), ..., v^N(x)\right)^T$ can be represented as follows

$$\mathbf{v}(x) = \mathcal{M}^{-1}\lambda,$$

where $\mathcal{M} = \Big(\Omega_j(\partial_k)_{j,k=1,..N}\Big), \quad \lambda = -\Upsilon\,(0, .., 0, \lambda_{M+1}, ..\lambda_N)$. or, what is the same,

$$\mathbf{v} = \sum_{j=M+1}^{N} \lambda_j X_j \tag{2.1}$$

where X_j, $j = M+1, ..., N$ constitute a maximal set of independent vector fields on $\mathcal{Q}$ satisfying the constraints, in the sense that the components of X_j satisfy the equations

$$\Omega_j(X_k) = 0, \quad j = 1, .., M, \, k = M+1, .., N$$

3. Let σ be the 1-form associated with the vector field $\mathbf{v}$, i.e.,

$$\sigma = (\mathbf{v}(x), dx) \equiv \sum_{j,k=1}^{N} G_{jk}(x) v^j(x) dx^k \equiv \sum_{k=1}^{N} p_k dx^k$$

then the 2-form $d\sigma$:

$$d\sigma = \frac{1}{2} \sum_{j,k=1}^{N} a_{jk}(x) \Omega_j \wedge \Omega_k,$$

where $A = (a_{jk})$ is a matrix such that

$$a_{jk} = (-1)^{j+k-1}\frac{1}{\Upsilon}d\sigma \wedge \Omega_1 \wedge ... \wedge \widehat{\Omega}_k .... \wedge \widehat{\Omega}_j .... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N)$$

$\widehat{\Omega}_j$, $\widehat{\Omega}_k$ means that these elements are omitted.

It is clear that the contraction of $d\sigma$ along $\mathbf{v}$ is

$$\iota_{\mathbf{v}} d\sigma = \sum_{j=1}^{N} \Lambda_j \Omega_j,$$

where

$$\Lambda \equiv col(\Lambda_1, \Lambda_2, ..., \Lambda_N) = A^T \lambda.$$

We shall analyze the differential equations

$$\dot{\mathbf{x}} = \mathcal{M}^{-1}\lambda \tag{2.2}$$

under the conditions

$$\begin{cases} \Lambda_j = 0, \quad j = M+1, .., N \\ \Upsilon = \Omega_1 \wedge \Omega_2 .... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N) \neq 0 \end{cases} \tag{2.3}$$

Corollary 2.1

For the case when $M = N - 1$ the vector field $\mathbf{v}$ takes the form

$$\mathbf{v} = \lambda_N det \begin{pmatrix} \Omega_1(\partial_1) & \Omega_1(\partial_2) & \dots & \Omega_1(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ \Omega_M(\partial_1) & \Omega_M(\partial_2) & \dots & \Omega_M(\partial_N) \\ \Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \dots & \Omega_{M+1}(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ \Omega_{N-1}(\partial_1) & \Omega_{N-1}(\partial_2) & \dots & \Omega_{N-1}(\partial_{N-1}) \\ \partial_1 & \partial_2 & \dots & \partial_N \end{pmatrix},$$

where λ_N is an arbitrary function.

The conditions (2.3) hold iff

$$\Upsilon = \Omega_1 \wedge \Omega_2 .... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N) \neq 0$$

where Ω_N is an arbitrary 1-form.

Proposition 2.1 *The differential equations*

$$\dot{\mathbf{x}} = \mathbf{v}(x), \quad x \in X$$

are invariant relationship of the Lagrangian equations with Lagrangian function

$$L_0 = \frac{1}{2}||\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})||^2$$

In fact, by derivation we deduce that $\nabla_{\dot{\mathbf{x}}}(\dot{\mathbf{x}} - \mathbf{v}(x)) = 0$, or,

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{\mathbf{x}}} L_0) = 0,$$

which are equivalent to Lagrangian equations with the Lagrangian function L_0 given above. It is easy to show that these equations admits the representation

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{x}^j} T) = \omega(\partial_j) + \nabla_{\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})} p_j$$

where

$$T = \frac{1}{2}||\dot{\mathbf{x}}||^2,$$
$$\omega = d\frac{||\mathbf{v}||^2}{2} + \iota_{\mathbf{v}} d\sigma,$$

σ is the 1- form associated with the vector field $\mathbf{v}$, $p_j = \sigma(\partial_j)$.

We shall study the case when (2.2) and (2.3) hold. The differential equations which describe the behavior of such mechanical systems under these restrictions can be represented as follows

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{x}^k} T) = \frac{\partial \frac{1}{2}||\mathbf{v}||^2}{\partial x^k} + \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_k). \tag{2.4}$$

and can be interpreted as the equations of motion of nonholonomic mechanical systems with an active potential field of force with potential U :

$$U = \frac{1}{2}||\mathbf{v}(x)||^2 + U_0, \quad U_0 = const.$$

and with the reactive forces with the components

$$(\sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_1), \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_2), ..., \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_N)),$$

generated by the constraints

$$\Omega_j(\dot{\mathbf{x}}) \equiv \sum_{k=1}^{N} \Omega_j(\partial_k) \dot{x}^k = 0.$$

It is interesting to observe that the solutions of the equations (2.4) depend on $2N - M$ initial conditions and solutions of the system (2.2) depend of N initial conditions and $N - M$ arbitrary functions which are solutions of the partial differential equations (2.3).

Corollary 2.2

If

$$\begin{cases} M = N - 1 \\ \Omega_j = df_j(x), \quad j = 1, 2, ..., N - 1 \\ \Omega_N = df_N \end{cases}$$

Then the equations (2.2)+(2.3) and (2.4) take the form respectively

$$\begin{cases} \dot{\mathbf{x}} = \lambda_N \det \begin{pmatrix} df_1(\partial_1) & \dots & df_1(\partial_N) \\ \vdots & & \vdots \\ df_{N-1}(\partial_1) & \dots & df_{N-1}(\partial_N) \\ \partial_1 & \dots & \partial_N \end{pmatrix} \equiv \lambda\{f_1,\, f_2, ..., f_{N-1}, *\} \\ \Upsilon = df_1 \wedge df_2 .... \wedge df_N(\partial_1,\, \partial_2,\, ..., \partial_N) \neq 0 \end{cases} \tag{2.5}$$

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{x}^k} T) = \frac{\partial \frac{1}{2}||\mathbf{v}||^2}{\partial x^k} + \sum_{j=1}^{N-1} \lambda_N a_{Nj}(x) df_j(\partial_k). \tag{2.6}$$

where λ_N is an arbitrary function:

$$df_N(\mathbf{v}) = -\Upsilon \lambda_N.$$

The studying of the behavior of the nonholonomic systems by using the equations (2.2)+(2.3) or (2.4) we called Cartesian and Lagrangian approach respectively [Sad, Ram].

Remark

For the system

$$\dot{\mathbf{x}} = \sum_{j=M+1}^{N} \lambda_j(t) X_j(x), \tag{2.7}$$

we can pose the optimal control problem [Bloch]

$$\min_{\lambda(\cdot)} \int_0^T \frac{1}{2} \sum_{j=M+1}^{N} \lambda_j^2(t) dt$$

subject to the dynamics (3.5)and the endpoint condition

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(T) = \mathbf{x}_1 \tag{2.8}$$

This optimal control problem is equivalent to the following variational problem (when certain assumptions hold) [Bloch]:

$$\min_{\lambda(\cdot)} \int_0^T \frac{1}{2} \sum_{j,k=1}^{N} B_{jk}\dot{x}^j \dot{x}^k dt \tag{2.9}$$

where $B == (B_{jk}) = MM^T$, and subject to the endpoint conditions (2.8) and

$$\Omega_j(\dot{\mathbf{x}}) = 0, \quad j = 1, 2, ...M. \tag{2.10}$$

As we can observe from the definition of the vector field $\mathbf{v}$ the function λ_j, $j = M+1, .., N$ are arbitrary smooth function and the 1-form Ω_j, $j = M+1, .., N$ are restricted only to the condition that $\Upsilon \neq 0$.

The arbitrariness can be restricted by using the complementary conditions. The functions $\lambda_{M+1}, .., \lambda_N$ can determine, for example, from the conditions (2.3), or from the optimal control problem (2.8)+(2.9)+(2.10).

With respect to the arbitrary 1-form we posed and study the following problem.

Problem

Determine the 1-form $\Omega_{M+1}, ...\Omega_N$ from the condition that the smallest Lie algebra of vector fields on $\mathcal{Q}$ that contains the vector field $X_{M+1}, ..., X_N$ is finite dimensional.

If we assume that

$$X_1, X_2, .., X_M, X_{M+1}, .., X_N, ..., X_S$$

is a basis of this Lie algebra then

$$[X_j,\, X_k] = \sum_{m=1}^{S} C_{jk}^m X_m, \quad j, k = 1, 2, ..S$$

where $X_j \in \xi(\mathcal{Q})$, $j = 1, 2, ..S$ and $[X, Y]$ is the Lie brackets of vector field X and Y, and C_{jk}^m are the structure constants.

When the algebra is three dimensional then from the Bianchi representation we obtain:

$$\begin{cases} [X_1, X_2] = aX_2 + b_3X_3 \\ [X_2, X_3] = b_1X_1 \\ [X_3, X_1] = b_2X_2 - aX_3 \end{cases} \tag{2.11}$$

where a, b_1, b_2, b_3 are certain constants

3. CARTESIAN APPROACH FOR NONHOLONOMIC SYSTEM WITH THREE DEGREE OF FREEDOM AND ONE CONSTRAINTS .

The case when $dim\mathcal{Q} = 3$ and $M = 1$ is of specific interest. We consider a natural mechanical system with configuration space $\mathcal{Q}$ and kinetic energy

$$T = \frac{1}{2} \sum_{k,j=1}^{3} G_{kj}(x)\dot{x}^j \dot{x}^k$$

Obviously, in this case the 1-form $\iota_{\mathbf{v}} d\sigma$ can be represented as follow

$$\iota_{\mathbf{v}} d\sigma = \Lambda_1 \Omega_1 + \Lambda_2 \Omega_2 + \Lambda_3 \Omega_3$$

where Λ_j, $j = 1, 2, 3$:

$$\begin{cases} \Lambda_1 = \Omega_2 \wedge \Omega_3(\mathbf{v}, rot\mathbf{v}) \\ \Lambda_2 = \lambda_3 \Omega_1(rot\mathbf{v}) \\ \Lambda_3 = -\lambda_2 \Omega_1(rot\mathbf{v}) \\ rot\mathbf{v} = \dfrac{1}{\sqrt{detG}} \left((\partial_y p_3 - \partial_z p_2)\partial_x + (\partial_z p_1 - \partial_x p_3)\partial_y + (\partial_x p_2 - \partial_y p_1)\partial_z, \right) \\ p_k = \displaystyle\sum_{j=1}^{3} G_{kj} v^j, \; k = 1, 2, 3 \end{cases}$$

The system (2.2)+(2.3) take the form respectively

$$\dot{\mathbf{x}} = [\Omega_1, \times \lambda_2 \Omega_3 - \lambda_3 \Omega_2] \equiv \lambda_2 X_2 + \lambda_3 X_3 \tag{3.1}$$

$$\begin{cases} \Upsilon \neq 0 \\ \Omega_1(rot\mathbf{v}) = 0 \end{cases} \tag{3.2}$$

where $\Omega_j(x) = (\Omega_j(\partial_x), \Omega_j(\partial_x)\Omega_j(\partial_x))$, $j = 1, 2, 3$; $[\quad \times, \quad]$ is the vector product on R^3.

Definition

The vector field $\mathbf{v}$ we call the Kummer vector field if

$$rot\mathbf{v} = \nu(x)\mathbf{v}$$

for certain function ν.

It is easy to show that the equations of motion (2.4) for $N = 3$ under the condition that $\mathbf{v}$ is a Kummer a vector field can be represented in Lagrangian form.

We shall determine the arbitrary 1-form Ω_2, Ω_3 in such form that the smallest Lie algebra of vector fields on $\mathcal{Q}$ that contains the vector field X_2, X_3 is finite dimensional.

In the assertions below, we illustrate the above results in concrete examples.

Let us suppose that

$$\Omega_1 = \epsilon dx + a(x, y, z)dy + b(x, y, z)dz, \quad \epsilon = const.$$

We choose Ω_2, Ω_3 as follow:

1. If $\epsilon = 1$ then

$$\Omega_2 = dx, \quad \Omega_3 = dy$$

2. If $a^2+b^2\neq 0, \forall (x,y,z)\in \mathcal{Q}$, then

$$\Omega_2=-bdy+adz,\quad \Omega_3=dx$$

The Cartesian vector field in this case takes the form respectively:

$$\begin{cases}\hat{\mathbf{v}}=\hat{\lambda}_2(a\partial_x-\partial_y)+\hat{\lambda}_3(b\partial_x-\partial_z)=\hat{\lambda}_2\hat{X}_2+\hat{\lambda}_3\hat{X}_3\\ \mathbf{v}=\lambda_2((a^2+b^2)\partial_x-\epsilon a\partial_y-\epsilon b\partial_z)+\lambda_3(b\partial_y-a\partial_z)\equiv\lambda_2X_2+\lambda_3X_3\end{cases}$$

The conditions (3.2) for the vector field $\hat{\mathbf{v}}$ in this case can be rewritten as follow

$$\begin{aligned}&\Upsilon=1\\ &\Omega_1(rot\,\tilde{v})=\partial_y\hat{\lambda}_2-\partial_z\hat{\lambda}_1+b(\partial_y(a\hat{\lambda}_1+b\hat{\lambda}_2)-a(\partial_z(a\hat{\lambda}_1+b\hat{\lambda}_2)+\partial_x\hat{\lambda}_2)=0,\end{aligned}\tag{3.3}$$

Clearly, between X_2, X_3 and $\hat{X}_2$, $\hat{X}_3$ there are the following relations

$$\begin{cases}X_2=a\hat{X}_2+b\hat{X}_3\\ X_3=a\hat{X}_3-b\hat{X}_2\end{cases}$$

Example: The Heisenberg system

We consider the following standard kinetic energy Lagrangian on Euclidean three dimensional space R^3 :

$$L=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)\tag{3.4}$$

subject to the constraint

$$\Omega_1(\dot{\mathbf{x}})=\dot{x}+z\dot{y}-y\dot{z}=0$$

The vector field $\mathbf{v}$ and $\hat{\mathbf{v}}$ in this case are such that:

$$\mathbf{v}=\lambda_2((y^2+z^2)\partial_x-z\partial_y+y\partial_y)-\lambda_1(y\partial_y+z\partial_z)=\lambda_2X_2+(-\lambda_3)X_3,\quad \Upsilon=y^2+z^2$$
$$\hat{\mathbf{v}}=\hat{\lambda}_2(z\partial_x-\partial_y)+\hat{\lambda}_3(-y\partial_x-\partial_z)=\hat{\lambda}_2\hat{X}_2-\hat{\lambda}_3\hat{X}_3,\quad \Upsilon=1$$

The vector field X_2, X_3 and $\hat{X}_2$, $\hat{X}_3$ are contains in the three dimensional Lie algebra with basis respectively

$$\begin{cases}X_2=(y^2+z^2)\partial_x-z\partial_y+y\partial_z\\ X_3=y\partial_y+z\partial_z\\ X_1=z\partial_y-y\partial_z\\ [X_2,X_3]=2X_1+2X_2,\quad [X_2,X_1]=-X_3,\quad [X_1,X_3]=0\end{cases}$$

and

$$\begin{cases} \hat{X}_2 = z\partial_x - \partial_y \\ \hat{X}_3 = y\partial_x - \partial_z \\ \hat{X}_1 = \partial_x \\ [\hat{X}_2, \hat{X}_3] = -2\hat{X}_1, \quad [\hat{X}_1, \hat{X}_3] = [\hat{X}_1, \hat{X}_2] = 0 \end{cases}$$

Which corresponds to the case in the Bianchi classification

$$a = b_3 = b_2 = 0, \quad b_1 = -2.$$

This algebra is well known as Heisenberg algebra.

To determine the function $\hat{\lambda}_2$, $\hat{\lambda}_3$ and λ_2, λ_3 we need to solve the equation respectively:

$$(1+y^2)\partial_y\hat{\lambda}_3 + yz\partial_z\hat{\lambda}_3 - y(\partial_x\hat{\lambda}_3 + y\hat{\lambda}_3) - ((1+z^2)\partial_z\hat{\lambda}_2 + yz\partial_y\hat{\lambda}_2 + z(\partial_x\hat{\lambda}_2 + z\hat{\lambda}_2)) = 0$$
$$(1+y^2+z^2)(y\partial_y\lambda_2 + z\partial_z\lambda_2 + 2\lambda_2) + y\partial_z\lambda_3 - z\partial_y\lambda_3 - (z^2+y^2)\partial_x\lambda_3 = 0$$

Clearly, the functions

$$\lambda_2 = \frac{\Psi(\frac{y}{z})}{z^2+y^2}, \quad \lambda_3 = \Phi(\sqrt{z^2+y^2})$$

where Ψ and Φ are arbitrary functions, are solutions of the second equation.

The solutions of the differential equation generated by $\mathbf{v}$ are

$$\begin{cases} \displaystyle\int \frac{dr}{\Phi(r)} = -t + t_0 \\ \displaystyle\int \frac{d\varphi}{\Psi(\tan\varphi)} = t - t_0 \\ x = x_0 + \varphi \\ y = r\cos\varphi \\ z = r\sin\varphi \end{cases}$$

Clearly, the initial conditions of these trajectories must be such that $(x_0, y_0, z_0) \neq (x_0, 0, 0)$.

Example: A nonholonomically Constrained Particle in R^3.

Consider a particle with the Lagrangian (3.4) and nonholonomic constraints

$$\Omega_1(\dot{\mathbf{x}}) \equiv \dot{x} + a(z)\dot{y} = 0$$

This instructive academic example, in the particular case when $a(z) = z$, due to Rosenberg [Ros]. This example was also used to illustrate the theory in Bates and Sniatycki [Bates]. The Cartesian approach in this case produce the following vector field $\hat{\mathbf{v}}$:

$$\hat{\mathbf{v}} = \lambda_2(a(z)\partial_x - \partial_y) - \lambda_3\partial_z$$

and condition (3.3) for this case takes the form

$$\lambda_2\Omega_1(rot\tilde{\mathbf{v}}) = \frac{1}{2}\partial_z((1+a^2)\lambda_2^2) + (a\partial_x\lambda_3 - \partial_y\lambda_3)\lambda_2 = 0. \tag{3.5}$$

The vector field X_2 and X_3 are such that

$$\begin{cases} X_2 = a(z)\partial_x - \partial_y \\ X_3 = \partial_z \end{cases}$$

$$[\tilde{X}_3, \tilde{X}_2] = \partial_z a(z)\partial_x$$

Clearly, we have a three dimensional Lie algebra if

$$a(z) = z$$
$$a(z) = \exp z$$

As we can observe in these cases we have respectively

$$\begin{cases} [\tilde{X}_2, \tilde{X}_3] = \tilde{X}_1 \equiv \partial_x \\ [\tilde{X}_3, \tilde{X}_1] = 0 \\ [\tilde{X}_2, \tilde{X}_1] = 0, \end{cases}$$

$$\begin{cases} [\tilde{X}_2, \tilde{X}_3] = -\tilde{X}_2 + \tilde{X}_1, \quad \tilde{X}_1 = \partial_y \\ [\tilde{X}_3, \tilde{X}_1] = 0 \\ [\tilde{X}_2, \tilde{X}_1] = 0, \end{cases}$$

By introducing the vector fields:

$$X_2 = \tilde{X}_2, \quad X_3 = \tilde{X}_3, \quad X_1 = \tilde{X}_1$$

and

$$X_1 = \tilde{X}_3, \quad X_3 = -\tilde{X}_2 + \tilde{X}_1, \quad X_2 = \tilde{X}_1$$

we deduce the Bianchi representation with $b_1 = 1$, $a = b_2 = b_3 = 0$ and $b_1 = a = b_2 = 0$, $b_3 = 1$.

If $a(z) = \cos z$ then by introducing the vector fields

$$\begin{cases} Y_1 = \cos z\partial_x - \partial_y \\ Y_2 = \partial_z \\ Y_3 = -\sin z\partial_x - \partial_y \\ Y_4 = \partial_y \end{cases}$$

we deduced the four-dimensional Lie algebra:

$$\begin{cases} [Y_1, Y_2] = Y_2 + Y_4, \\ [Y_2, Y_3] = -Y_1 - Y_4 \\ [Y_1, Y_3] = 0 \\ [Y_1, Y_4] = 0 \\ [Y_2, Y_4] = 0 \\ [Y_3, Y_4] = 0 \end{cases}$$

We shall study the case when in (3.5)

$$\lambda_2 = \frac{A}{\sqrt{a^2+1}}, \quad \lambda_3 = b_2(z),$$

for A an arbitrary constant and b_2 an arbitrary function.

The equations generated by the vector field $\hat{\mathbf{v}}$ in this case are

$$\begin{cases} \dot{x} = \dfrac{a(z)A}{\sqrt{1+a^2(z)}} \\ \dot{y} = -\dfrac{A}{\sqrt{1+a^2(z)}} \\ \dot{z} = -b_2(z) \end{cases}$$

Hence the all trajectories of these equations are the following

$$\begin{cases} x = x_0 - A\displaystyle\int_0^z \frac{a(z)dz}{b_2(z)\sqrt{1+a^2(z)}} \\ y = y_0 - A\displaystyle\int_0^z \frac{dz}{b_2(z)\sqrt{1+a^2(z)}} \\ t = t_0 - \displaystyle\int_0^z \frac{dz}{b_2(z)} \end{cases} \tag{3.6}$$

The equation (2.4) in this case may be rewritten as

$$\begin{cases} \ddot{x} = \partial_z(\dfrac{Aa(z)}{\sqrt{a^2(z)+1}}) \\ \ddot{y} = a(z)\partial_z(\dfrac{Aa(z)}{\sqrt{a^2(z)+1}}) \\ \ddot{z} = \partial_z b_2(z) \end{cases}$$

Corollary 3.1 All the trajectories of the equation of motion of the constrained Lagrangian system

$$< \mathrm{R}^3,\, L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(z),\, \{\dot{x} + a(z)\dot{y} = 0\} >$$

can be obtained from (3.6) [Sad].

Example

The trajectories of the constrained Lagrangian system

$$< \mathrm{R}^3,\, L=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2),\, \{\dot{x}+\tan z^2\dot{y}=0\} >$$

are the following

$$\begin{cases} x=x_0-BA\displaystyle\int_0^z \cos z^2 dz = x_0-ABC(z) \\ y=y_0-BA\displaystyle\int_0^z \sin z^2 dz = y_0-ABS(z) \\ t=t_0-Bz \end{cases}$$

Analogously we can show that the trajectories of the constrained Lagrangian system

$$< \mathrm{R}^3,\, L=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)-\frac{z}{B^2},\, \{\dot{x}+\tan z\dot{y}=0\} >$$

are

$$\begin{cases} x=x_0-BA\displaystyle\int_0^z \frac{\cos z dz}{\sqrt{z}} = x_0-ABC(z) \\ y=y_0-BA\displaystyle\int_0^z \frac{\sin z dz}{\sqrt{z}} = y_0-ABS(z) \\ t=t_0-B\sqrt{z} \end{cases}$$

where A, B, t_0, x_0, y_0 are constants, $C(z)$ and $S(z)$ are the Fresnel integrals.

The Fresnel integrals has important applications in the physics of diffraction and is used in the theory of driving motorcar round a corner quickly.

The Cornu spiral, also known as clothoid, is the curve generated by a parametric plot of S(x) against C(x). The Cornu spiral was created by Marie Alfred Cornu as a nomogram for diffraction computations in science and engineering.

Example: The Chapliguin-Caratheodory sleigh

We shall now analyse one of the most classical nonholonomic systems : Chapliguin-Carathodory's sleigh [NF],[Poliajov]. The idealized sleigh is a body that has three points of contact with the plane. Two of them slide freely but the third, A, behaves like a knife edge subjected to a constraining force $\mathbf{R}$ which does not allow transversal velocity. More precisely, let yoz be an inertial frame and $\xi\, A\eta$ a frame moving with the sleigh. Take as generalized coordinates the Cartesian coordinates of the center of mass C of the sleigh and the angle x between the y and the ξ axis. The reaction force $\mathbf{R}$ against the runners is exerted laterally at the point of application A in such a way that the η component of the velocity is zero. Hence, one has the constrained system $\mathcal{M}$ with the configuration space $X = S^1 \times \mathrm{R}^2$, with the kinetic energy $T=\frac{m}{2}(\dot{y}^2+\dot{z}^2)+\frac{I_c}{2}\dot{x}^2$, and with the constraint $\epsilon\dot{x}+\sin x\dot{y}-\cos x\dot{z}=0$, where m is the mass of the system and I_c is the moment of inertia about a vertical axis through C and $\epsilon = |AC|$. Observe that the "javelin" (or arrow or Chapliguin's skate) is a particular case of this mechanical system and can be obtained when $\epsilon = 0$

To apply the Cartesian approach for this system, first we introduce the 1-form Ω_j, $j = 1, 2, 3$ in such a way that the determinant $\Upsilon \neq 0$. In this subcase, we achieve this condition if

$$\Omega_1 = \epsilon dx + \sin x dy - \cos x dz, \quad \Omega_2 = \sin x dz + \cos x dy, \quad \Omega_3 = dx.$$

Under these restrictions we obtain that $\Upsilon = 1$.

The vector field $\mathbf{v}$ takes the form

$$\mathbf{v} = \lambda_3(\partial_x - \epsilon \sin x \partial_y - \epsilon \cos x \partial_z) - \lambda_2(\cos x \partial_y - \sin x \partial_z) = \lambda_3 X_3 + \lambda_2 X_2$$

Let X_1, X_2, X_3 be the vector field:

$$\begin{cases} X_3 = \partial_x - \epsilon \sin x \partial_y - \epsilon \cos x \partial_z \\ X_2 = \cos x \partial_y - \sin x \partial_z \\ X_1 = \partial_x \\ [X_2, X_3] = -[X_1, X_3] = \epsilon X_2, \quad [X_1, X_2] = -\dfrac{1}{\epsilon}(X_3 - X_1) \end{cases}$$

if $\epsilon \neq 0$.

Then if we introduce the vector fields respectively:

$$Z = \epsilon X_2, \quad X = X_1, \quad Y = X_1 - X_3$$

we deduce the Bianchi representation with $a = b_1 = 0$, $b_3 = -b_2 = 1$.

and

$$\begin{cases} X_3 = \partial_x \\ X_2 = \cos x \partial_y - \sin x \partial_z \\ X_1 = \cos x \partial_z + \sin x \partial_y \\ [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = 0 \end{cases}$$

if $\epsilon = 0$, then we obtain the Bianchi representation with $b_1 = b_2 = 1$, $a = b_3 = 0$.

The conditions $\Omega_1(rot\mathbf{v}) = 0$ in this case holds if

$$\begin{cases} \partial_y \lambda_3 = \dfrac{\epsilon m}{J_C + \epsilon^2 m} \partial_z \lambda_2 \\ \partial_z \lambda_3 = -\dfrac{\epsilon m}{J_C + \epsilon^2 m} \partial_y \lambda_2 \\ \partial_x \lambda_2 = \epsilon \lambda_3 \end{cases}$$

After some calculations we can prove that the solutions of theses equations are

$$\begin{cases} \lambda_2 = \cos \alpha \, V_1(y, z, \epsilon) + \sin \alpha \, V_2(y, z, \epsilon) + \epsilon \mu(x, \epsilon) \\ \lambda_3 = \dfrac{\epsilon m}{J_C + \epsilon^2 m} \Big(cos\alpha \, V_2(y, z, \epsilon) - \sin \alpha \, V_1(y, z, \epsilon) \Big) + \dfrac{d\mu}{dx}(x, \epsilon), \\ \alpha = \dfrac{\epsilon^2 m x}{J_C + \epsilon^2 m} \end{cases}$$

where K is an arbitrary function and V_1, V_2 are functions which satisfy the Cauchy-Riemann conditions:

$$\begin{cases} \partial_y V_1(y,z,\epsilon) = \partial_z V_2(y,z,\epsilon) \\ \partial_z V_1(y,z,\epsilon) = -\partial_y V_2(y,z,\epsilon). \end{cases} \tag{3.7}$$

The aim of the assertions below is to study the differential equations generated by the vector field $\mathbf{v}$:

$$\begin{cases} \dot{x} = \lambda_3(x,y,z,\epsilon) \\ \dot{y} = \lambda_2(x,y,z,\epsilon)\cos x - \epsilon\lambda_3 \sin x \\ \dot{z} = -\lambda_2(x,y,z,\epsilon)\sin x - \epsilon\lambda_3\cos x \end{cases} \tag{3.8}$$

For $\epsilon \neq 0$ after the change

$$\begin{cases} x = x, \quad y = \epsilon Y, \quad z = \epsilon Z \\ V_1 = \epsilon v(Y,Z,\epsilon), \quad V_2 = \epsilon u(Y,Z,\epsilon) \end{cases}$$

the above equations can be rewritten as follow

$$\begin{cases} \dot{x} = -\dfrac{d\mu}{dx} + \nu(u\sin\alpha - v\cos\alpha) \\ \dot{Y} = \sin x\dfrac{d\mu}{dx} - \cos x\mu - u(\cos\alpha\cos x + \nu\sin\alpha\sin x) - v(\sin\alpha\cos x - \nu\cos\alpha\sin x) \\ \dot{Z} = -\sin x\mu - \cos x\dfrac{d\mu}{dx} - v(\sin\alpha\sin x + \nu\cos\alpha\cos x) - u(\sin x\cos\alpha - \nu\sin\alpha\cos x) \\ \nu = \dfrac{\epsilon^2 m}{J_C + \epsilon^2 m}, \quad \alpha = \nu x. \end{cases}$$

Corollary 3.2
Let us introduce the notation $\mathbf{v}_\infty = \lim_{\epsilon\to\infty} \mathbf{v}$ Then the equations generated by the vector field $\mathbf{v}_\infty$ are

$$\begin{cases} \dot{x} = -\dfrac{d\mu}{dx}(x,\infty) - \Re(F(w)\exp(\imath x)), \quad w = Y + \imath Z \\ \dot{w} + F(w) = (\mu(x,\infty) + \imath\dfrac{d\mu}{dx}(x,\infty))\exp(\imath x) \end{cases}$$

where $F(w) = u + \imath v$ is an holomorphic function on $D \subset \mathrm{C}$ In particular if $\mu = 0$ the above equations admits the first integral

$$H(Y,Z) = C.$$

where H is a function:

$$u(Y,Z,\infty) = -\partial_Z H(Y,Z), \quad v(Y,Z,\infty) = \partial_Y H(Y,Z)$$

Corollary 3.3

Let us suppose that

$$\lambda_j = \lambda_j(x, \epsilon), \quad j = 1, 2$$

then the all trajectories of the Chapliguin- Caratheodory sleigh under the action of the potential field of force with potential U :

$$2(U + h) = m\lambda_2^2(x, \epsilon) + (m\epsilon^2 + J_C)\lambda_3^2(x, \epsilon)$$

can be obtained from the formula

$$\begin{cases} y = y_0 + \displaystyle\int_{x_0}^{x} \frac{(\lambda_2(x, \epsilon)\cos x - \epsilon\lambda_3 \sin x)dx}{\lambda_3(x, \epsilon)} \\ z = z_0 - \displaystyle\int_{x_0}^{x} \frac{(\lambda_2(x, y, z, \epsilon)\sin x - \epsilon\lambda_3 \cos x)dx}{\lambda_3(x, \epsilon)} \\ t = t_0 + \displaystyle\int_{x_0}^{x} \frac{dx}{\lambda_3(x, \epsilon)} \end{cases}$$

Clearly, if the movements are given by inertia, then

$$\begin{cases} \lambda_2 = C_1 \sin(\sqrt{\dfrac{m}{I_C + m\epsilon^2}}\epsilon x + C_2), \\ \lambda_3 = C_1 \sqrt{\dfrac{m}{I_C + m\epsilon^2}}\cos(\sqrt{\dfrac{m}{I_C + m\epsilon^2}}\epsilon x + C_2) \end{cases}$$

where C_1, C_2 are arbitrary constants.

Corollary 3.4

The all trajectories of the Chapliguin skate ($\epsilon = 0$) and the Chapliguin- Caratheodory sleigh by inertia ($K = 0, \quad V_1 = A, \quad V_2 = B,\ A, B = const.$) can be obtained from the Cartesian approach.

It is easy to prove that for $\epsilon = 0$ the following relation holds

$$rot\mathbf{v} = -m\mathbf{v} - m\Big(K(x, 0) + \partial_y V_2(y, z, 0)\sin x - \partial_z V_2(y, z, 0)\cos x\Big)\partial_x,$$

As a consequence we give that the vector field $\mathbf{v}$:

$$\begin{cases} \dot{x} = -(b\cos x - a\sin x) \\ \dot{y} = -(ay + bz + c)\cos x \\ \dot{z} = -(ay + bz + c)\sin x \end{cases}$$

is a Kummer vector field.

Example : Heavy rigid body in the Suslov case

In this section we study one classical problem of non-holonomic dynamics formulated by Suslov [Suslov]. In this problem we consider the rotational motion of a rigid body around a fixed point. The motion of the body is restricted by one non-holonomic constraints $(\mathbf{a}, \omega) = 0$ where ω is a body angular velocity.

The nonholonomic equations of motion we take in the form

$$\begin{cases} I\dot{\omega} = [I\omega, \omega] + [\gamma, \dfrac{\partial U}{\partial \gamma}] + [\Lambda\omega, \gamma] + \mu\mathbf{a} \\ \dot{\gamma} = [\gamma, \omega] \\ (\mathbf{a}, \omega) = 0 \end{cases}$$

Where

$$\begin{aligned} & I = diag(I_1, I_2, I_3), \\ & \Lambda = diag(\lambda_1,\, \lambda_2,\, \lambda_3), \quad l_j = consts. \\ & \gamma = (\gamma_1 = \sin z \sin x, \quad \gamma_2 = \sin z \cos x, \quad \gamma_3 = \cos z) \end{aligned}$$

I_1, I_2, I_3 are the inertial moment of the body.

If we assume that the vector $\mathbf{a} = (0, 0, 1)$ [Koz2], then

$$\begin{cases} I_1\omega_1 - \lambda_2\gamma_1 \equiv \Phi_1 \\ I_2\omega_2 - \lambda_1\gamma_2 \equiv \Phi_2 \\ \dot{\Phi}_1 = \gamma_3\partial_{\gamma_2}U - \gamma_2\partial_{\gamma_3}U \\ \dot{\Phi}_2 = \gamma_1\partial_{\gamma_3}U - \gamma_3\partial_{\gamma_1}U \\ (I_1 - I_2)\omega_1\omega_2 + \gamma_2\partial_{\gamma_2}U - \gamma_2\partial_{\gamma_1}U + \mu = 0 \\ \dot{\gamma}_1 = -\gamma_3\omega_2 \\ \dot{\gamma}_2 = \gamma_3\omega_1 \\ \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1 \end{cases} \tag{3.9}$$

The above system has two independent first integrals

$$\begin{aligned} K_1 &= \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2) - U(\gamma_1, \gamma_2, \gamma_3) \\ K_2 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \end{aligned}$$

By the Jacobi's theorem about the last multiplier, if there exits a third independent first integral K_3 which is functionally independent together with K_1 and K_2, then the Suslov problem is integrable by quadratures [Koz2, Mac]

The integrable cases of system (3.8) are the following:

Suslov case

If we suppose that $U = 0$ then

$$K_3 = I_1\omega_1 + \lambda_2\gamma_1, \quad K_4 = I_2\omega_2 + \lambda_1\gamma_2$$

Kharlamova-Zabelina case

If we suppose that

$$U = \Psi(C_1\gamma_1 + C_2\gamma_2)$$

where Ψ is an arbitrary smooth function, then

$$K_3 = I_1C_1\omega_1 + I_2C_2\omega_2 + (\lambda_2C_1\gamma_1 + \lambda_1C_2\gamma_2)$$

Kozlov case

If

$$U = \Psi(1 - \gamma_1^2 - \gamma_2^2), \quad I_1 = I_2, \quad \lambda_1 = \lambda_2 = 0$$

where Ψ is an arbitrary smooth function, then

$$K_3 = \omega_1\gamma_1 + \omega_2\gamma_2$$

Corollary 3.5

Let us suppose that the potential function U in (3.9)is determine as follows

$$U = \frac{1}{2I_1I_2}(I_1\mu_1^2 + I_2\mu_2^2) - h \tag{3.10}$$

where μ_1, $\mu_2$are solutions of the partial differential equations

$$\gamma_3\left(\frac{\partial\mu_1}{\partial\gamma_2} - \frac{\partial\mu_2}{\partial\gamma_1}\right) + \gamma_2\frac{\partial\mu_1}{\partial\gamma_3} - \gamma_1\frac{\partial\mu_2}{\partial\gamma_3} = 0 \tag{3.11}$$

Then the equations (3.9)+(3.10) admits the first integrals

$$K_3 = I_1\omega_1 + \lambda_2\gamma_1 - \mu_1, \quad K_4 = I_2\omega_2 + \lambda_1\gamma_2 + \mu_2 \tag{3.12}$$

The proof we obtain by considering that (3.9)+(3.10) can be rewritten as follows:

$$\begin{cases} \dot\Phi_1 = \dot\mu_2 + KI_1\mu_1 \\ \dot\Phi_2 = -\dot\mu_1 + KI_2\mu_2 \\ (I_1 - I_2)\omega_1\omega_2 + I_1\mu_1(\gamma_2\partial_{\gamma_1}\mu_1 - \gamma_1\partial_{\gamma_2}\mu_2) + \\ \qquad I_2\mu_2(\gamma_2\partial_{\gamma_1}\mu_2 - \gamma_1\partial_{\gamma_2}\mu_2) + \lambda_2\omega_2\gamma_1 - \lambda_1\omega_1\gamma_2 + \mu = 0 \\ K \equiv \gamma_3\left(\frac{\partial\mu_1}{\partial\gamma_2} - \frac{\partial\mu_2}{\partial\gamma_1}\right) + \gamma_2\frac{\partial\mu_1}{\partial\gamma_3} - \gamma_1\frac{\partial\mu_2}{\partial\gamma_3} \end{cases}$$

The aim of this apartat is to propose the Cartesian approach for this nonholonomic system.

Let us suppose that $\mathcal{Q} = SO(3)$, with the Riemann metric

$$G = \begin{pmatrix} I_3 & I_3\cos z & 0 \\ I_3\cos z & (I_1\sin^2 x + I_2\cos^2 x)\sin^2 z + I_3\cos^2 z & (I_1 - I_2)\sin x\cos x\sin z \\ 0 & (I_1 - I_2)\sin x\cos x\sin z & I_1\cos^2 x + I_2\sin^2 x \end{pmatrix}$$

$detG = I_1I_2I_3\sin^2 z,$

The given 1-form is the following $\Omega_1 = dx + \cos z dy$. By choosing the 1-form Ω_2, Ω_3 as follow

$$\Omega_2 = dy, \quad \Omega_3 = dz$$

we obtain that $\Upsilon = 1$

The equations (3.1) and conditions(3.2) take the form respectively

$$\begin{cases} \dot{x} = \cos z \, \lambda_2, \\ \dot{y} = -\lambda_2, \\ \dot{z} = -\lambda_3, \end{cases} \tag{3.13}$$

$$\Omega_1(rot\mathbf{v}) = \partial_z p_2 - \partial_y p_3 + \cos z \partial_x p_3 = 0 \tag{3.14}$$

where

$$\begin{cases} rot\mathbf{v} = \frac{1}{\sqrt{detG}}(\partial_z p_2 - \partial_y p_3, \, \partial_x p_3, \, -\partial_x p_2) \\ p_k = \frac{\partial T}{\partial \dot{x}^k}|_{\dot{\mathbf{x}}=\mathbf{v}} \end{cases} \tag{3.15}$$

By introducing the vector field X_1, X_2, X_3 :

$$\begin{cases} X_1 = -\sin z \partial_x - \partial_y \\ X_2 = \cos z \partial_x - \partial_y \\ X_3 = \partial_z \\ X_4 = \partial_y \end{cases}$$

we deduce the four dimensional algebra introduce in the example: A nonholonomically Constrained Particle in R^3.

From (3.15) we deduce that

$$\begin{cases} -I_1 I_2 \lambda_3 = I_1 p_3 + (I_2 - I_1)(p_3 \cos x + \frac{p_2 \sin x}{\sin z} \\ -I_1 I_2 \sin z \lambda_2 = -\frac{I_1 p_2}{\sin z} + (I_1 - I_2) \sin x (p_3 \cos x + \frac{p_2 \sin x}{\sin z}) \end{cases}$$

By introducing the change

$$\begin{cases} p_2 = \sin z(\mu_2 \sin x - \mu_1 \cos x) \\ p_3 = \mu_2 \cos x + \mu_1 \sin x \end{cases}$$

We obtain that the system (3.13)and equation (3.14) admit the representation respectively

$$\begin{cases} \dot{x} = \frac{\cot z}{I_1 I_2} (I_1 \mu_1 \cos x - I_2 \mu_2 \sin x), \\ \dot{y} = \frac{-1}{I_1 I_2 \sin z}(I_1 \mu_1 \cos x - I_2 \mu_2 \sin x) \\ \dot{z} = \frac{1}{I_1 I_2}(I_1 \mu_1 \sin x + I_2 \mu_2 \cos x), \end{cases} \tag{3.16}$$

$$\begin{aligned}\sin z\sin x\partial_z\mu_2+\cos z\cos x\partial_x\mu_2-\cos x\partial_y\mu_2+\\ \cos z\sin x\partial_x\mu_1-\sin z\cos x\partial_z\mu_1-=0\end{aligned} \tag{3.17}$$

Clearly,

$$||\mathbf{v}||^2=\frac{1}{I_1I_2}(I_1\mu_1^2+I_2\mu_2^2)$$

Denoting by $\gamma_1=\sin z\sin x,\quad \gamma_2=\sin z\cos x,\quad \gamma_3=\cos z$ from (3.16) and (3.17) we obtain the relations respectively

$$\begin{cases}\dot{\gamma_1}=\dfrac{I_1}{I_1I_2}\mu_1\gamma_3\\ \dot{\gamma_2}=\dfrac{I_2}{I_1I_2}\mu_2\gamma_3\\ \dot{\gamma_3}=\dfrac{-1}{I_1I_2}(I_1\mu_1\gamma_1+I_2\mu_2\gamma_2)\end{cases} \tag{3.18}$$

$$\cos z\sin z(\frac{\partial\mu_1}{\partial\gamma_2}-\frac{\partial\mu_2}{\partial\gamma_1})-\cos x\partial_y\mu_2-\sin x\partial_y\mu_1=0$$

We shall study the case when

$$\lambda_1=\lambda_2=0,\quad \mu_j=\mu_j(x,z),\, j=1,2$$

Hence,

$$\frac{\partial\mu_1}{\partial\gamma_2}-\frac{\partial\mu_2}{\partial\gamma_1} \tag{3.19}$$

by compare (3.18) with (3.9) we deduce that

$$I_1\omega_1=\mu_2,\quad I_2\omega_2=-\mu_1$$

In view of (3.19) we obtain

$$\begin{cases}\mu_1=\dfrac{\partial S}{\partial\gamma_1}\\ \mu_2=\dfrac{\partial S}{\partial\gamma_2}\\ S=S_0(\gamma_1\gamma_2\gamma_3)+S_1(\gamma_1,\gamma_2)\end{cases}$$

where S, S_0 are arbitrary smooth functions.

The following particular case are well known:

Suslov case [Koz].

If in the above relations we suppose that

$$S=C_1\gamma_1+C_2\gamma_2$$

or, what is the same

$$\mu_1 = C_1, \quad \mu_2 = C_2, \quad C_j = const., \, j = 1, 2$$

then we obtain the Suslov case.

Suslov-Kharlamova-Zabelina case.

If we suppose that

$$S = \frac{3}{2\sqrt{I_1 C_1^2 + I_2 C_2^2}} (\sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2})^3 + \frac{C}{2C_1 I_1}\gamma_1 - \frac{C}{2C_2 I_2}\gamma_2$$

where $\tilde{h}$, C_1, C_2, C are constants, then

$$\begin{cases} \mu_1 = \dfrac{C_1}{\sqrt{I_1 C_1^2 + I_2 C_2^2}} \sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} + \dfrac{C}{2C_1 I_1} \\ \mu_2 = \dfrac{C_2}{\sqrt{I_1 C_1^2 + I_2 C_2^2}} \sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} - \dfrac{C}{2C_2 I_2} \end{cases}$$

which correspond to the Suslov-Kharlamova-Zabelina case.

Now we shall study the case when (3.17) holds in view of the relations

$$\begin{cases} \tan z \partial_z \mu_2 = -\partial_x \mu_1 \\ \partial_x \mu_2 = \tan z \partial_z \mu_1 \end{cases} \tag{3.20}$$

From the compatibility conditions we obtain the following partial differential equation

$$\sin^2 z \cos z \frac{\partial^2 \mu_j}{\partial z \partial z} + \cos^3 z \frac{\partial^2 \mu_j}{\partial x \partial x} + \sin z \frac{\partial \mu_j}{\partial z} = 0, \quad j = 1, 2$$

Corollary 3.6

Let μ_1 and μ_2 are solutions of the system (3.20), then the function $F = \mu_2 + \imath\mu_1$ is holomorphic function on the complex variable $w = \gamma_2 + \imath\gamma_1$.

In fact, after the change $u = \ln \sin z$ from (3.20) we deduced the Cauchy-Riemann equations

$$\begin{cases} \partial_u \mu_2 = -\partial_x \mu_1 \\ \partial_x \mu_2 = \partial_u \mu_1 \end{cases}$$

i.e., the function

$$F(u + \imath x) = F(\ln w)$$

is an holomorphic function, hence

$$\mu_1 = \Re F, \quad \mu_2 = \Im F$$

Clearly, μ_1, μ_2 are solution of the Laplace equation

$$\frac{\partial^2 \mu_j}{\partial u \partial u} + \frac{\partial^2 \mu_j}{\partial x \partial x} = 0$$

Hence, if

$$\mu_j = X_j(x) Y_j(u), \quad j = 1, 2 \tag{3.21}$$

then X and Y are solution of the second ordinary differential equation respectively

$$X_j^{''}(x) + \nu^2 X_j(x) = 0, \tag{3.22}$$

and

$$Y_j^{''}(u) - \nu^2 Y_j(u) = 0, \quad j = 1, 2 \tag{3.23}$$

where ν is a real constant.

Corollary 3.7

Let μ_1 and μ_2 are solutions of the system (3.20), then its are solutions of the partial differential equations

$$\frac{\partial^2 \mu_j}{\partial u \partial u} + \frac{u^2+1}{u-u^3} \frac{\partial \mu_j}{\partial u} + \frac{u^2}{(u^2-1)^2} \frac{\partial^2 \mu_j}{\partial x \partial x} = 0, \quad u = \cos z, \quad j = 1, 2$$

If we represent μ_j by the formula (3.21) then X is a solution of the differential equation (3.22). and Y is a solution of the Fuchsian equation

$$Y_j''(u) + \frac{u^2+1}{u-u^3} Y_j'(u) - \frac{\nu^2 u^2}{(u^2-1)^2} Y_j(u) = 0, \quad j = 1, 2 \tag{3.24}$$

The proof we obtain after the calculations from (3.20), after the change $u = \cos z$.

Analogously we can prove the following assertion

Corollary 3.8

Let μ_1 and μ_2 are solutions of the system (3.20), then its are solutions of the partial differential equations

$$\frac{\partial^2 \mu_j}{\partial u \partial u} + \frac{1}{u} \frac{\partial \mu_j}{\partial u} + \frac{1}{u^2} \frac{\partial^2 \mu_j}{\partial x \partial x} = 0, \quad u = \sin z, \quad j = 1, 2$$

Hence, if (3.21) holds then X satisfies (3.22) and Y is a solution of the Euler ordinary differential equation

$$u^2 Y_j''(u) + u Y_j'(u) - Y_j(u) = 0, \quad j = 1, 2 \tag{3.25}$$

where ν is real constant.

Proposition 3.1
The functions

$$\begin{cases} K_3 = I_1\omega_1 - X_1(x)Y_1(u) \\ K_4 = I_2\omega_2 - X_2(x)Y_2(u) \end{cases}$$

where X_1, X_2 are solutions of (3.22) and Y_1, Y_2 are solutions of (3.23) or (3.24) or (3.25), are the first integrals of the system (3.9)+(3.10)+(3.11)+(3.19).

Corollary 3.9

Let us suppose that in (3.18) $I_1 = I_2$. Then if the conditions of corollary hold then these equations can be rewritten as follow

$$\begin{cases} \dot{\gamma_1} = \gamma_3 \partial_{\gamma_2} H(\gamma_1,\, \gamma_2) \\ \dot{\gamma_2} = -\gamma_3 \partial_{\gamma_1} H(\gamma_1,\, \gamma_2) \\ \dot{\gamma_3} = \gamma_2 \partial_{\gamma_1} H(\gamma_1,\, \gamma_2) - \gamma_1 \partial_{\gamma_2} H(\gamma_1,\, \gamma_2) \end{cases}$$

where H is a function:

$$\mu_1 = \partial_{\gamma_2} H(\gamma_1,\, \gamma_2), \quad \mu_2 = -\partial_{\gamma_1} H(\gamma_1,\, \gamma_2).$$

Hence we obtain the existence of the first integral

$$H(\gamma_1,\, \gamma_2) = C.$$

To conclude the construction the Cartesian approach for heavy rigid body in the Suslov case we analyze the case when **v** is a Kummer vector field.

In view of (3.15) we obtain that **v** is a Kummer vector field if

$$\begin{cases} \partial_z p_2 - \partial_y p_3 = 0 \\ \partial_x p_3 = \nu p_2 \\ \partial_x p_2 = -\nu p_3 \end{cases}$$

Hence we deduce that if $\nu = const. \neq 0$ then

$$\begin{cases} p_2 = a(y,z)\cos\nu x + b(y,z)\sin\nu x \\ p_3 = b(y,z)\cos\nu x - a(y,z)\sin\nu x \end{cases}$$

where a and b are function:

$$\partial_y a = \partial_z b, \quad \partial_z a = -\partial_y b,$$

and if $\nu = 0$ then

$$p_2 = \partial_y S(y,z), \quad p_3 = \partial_z S(y,z)$$

The relation between p_2, p_3 and μ_1, μ_2 are the following

$$\mu_1 = p_3 \sin x - \frac{p_2}{\sin z}\cos x, \quad \mu_2 = p_3 \cos x + \frac{p_2}{\sin z}\sin\, x.$$

It is easy to observe that if $I_1 = I_2$ then S is a first integral of $\mathbf{v}$.

4. CARTESIAN APPROACH FOR NONHOLONOMIC SYSTEM WITH FOUR DEGREE OF FREEDOM AND TWO CONSTRAINTS .

In this apartat we illustrate the Cartesian approach for mechanical system with four degree of freedom and two constraints in the well known example which we call the Gantmacher system [Gantmacher]

Example: Gantmacher system

Let

$$\mathcal{Q} = \mathrm{R}^4, \quad T = \frac{1}{2}\sum_{j=1}^{4}\dot{u}_j^2$$

be the configuration space and kinetic energy of the mechanical system which is under the constraints

$$\begin{cases} u_1\dot{u}_1 + u_2\dot{u}_2 = 0 \\ u_1\dot{u}_3 - u_2\dot{u}_4 = 0 \end{cases}$$

By choosing the 1-forms Ω_3, Ω_4 as follow

$$\Omega_3 = u_1 du_2 - u_2 du_1, \quad \Omega_4 = u_2 du_3 + u_1 du_4$$

we obtain that

$$\begin{aligned} &\Upsilon = (u_1^2 + u_2^2)^2 \\ &\mathbf{v} = \nu_3(u_1\partial_2 - u_2\partial_1) + \nu_4(u_2\partial_3 + u_1\partial_4) \equiv \nu_3 X_3 + \nu_4 X_2 \\ &\nu_j = \lambda_j(u_1^2 + u_2^2), \quad j = 1, 2 \end{aligned}$$

The conditions (2.3)

$$\Lambda_3 = \Lambda_4 = 0$$

hold iff

$$u_2\partial_1\nu_4 - u_1\partial_2\nu_4 + u_2\partial_3\nu_3 + u_1\partial_4\nu_3 = 0 \tag{4.1}$$

The vector fields X_3, X_4 are contain in three dimensional Lie algebra:

$$\begin{cases} [X_3, X_2] = u_1\partial_3 - u_2\partial_4 \equiv X_1 \\ [X_1, X_2] = 0, \quad [X_1, X_3] = -X_2 \end{cases}$$

which correspond to the Bianchi representation with $a = b_1 = 0$, $b_3 = -b_2 = 1$.

It is easy to show that the functions ν_3, ν_4 :

$$\nu_3 = g_3(u_1^2 + u_2^2), \qquad \nu_4 = \sqrt{\frac{2g(u_4 + h)}{(u_1^2 + u_2^2)} - g_3^2(u_1^2 + u_2^2)},$$

where g, h are constants, are solutions of (4.1).

The solutions of the differential equations generated by the vector field $\mathbf{v}$ in this case are

$$\begin{cases} u_1 = r\cos\alpha \\ u_2 = r\sin\alpha \\ u_3 = u_3^0 + \dfrac{g}{2g_3(r)}t - \dfrac{g}{4g_3^2(r)}\sin 2\alpha - -\dfrac{\sqrt{2g}C}{g_3(r)}\cos\alpha \\ u_4 = -h + \dfrac{r^2 g_3^2(r)}{2g} + \Big(\dfrac{\sqrt{g}}{\sqrt{2}g_3(r)}\sin\alpha + C\Big)^2 \\ \alpha = \alpha_0 + g_3(r)t \\ r = \sqrt{u_1^2 + u_2^2} \end{cases},$$

where C, r, α_0, u_3^0, h, are arbitrary constants, g_3 is an arbitrary on r function. These functions coincide with the solutions of the equations of motion obtained from the d'Alembert-Lagrange principle [Gantmacher].

The Cartesian approach in this case produces additional possible trajectories of the Gantmacher system. In fact, by introducing the polar coordinates

$$\begin{cases} u_1 = r_1\cos\varphi_1 \\ u_2 = r_1\sin\varphi_1 \\ u_3 = r_2\cos\varphi_2 \\ u_4 = r_2\sin\varphi_2 \end{cases}$$

we obtain that the differential equations generated by the vector field $\mathbf{v}$ take the form

$$\begin{cases} \dot{r}_1 = 0 \\ \dot{\varphi}_1 = \nu_3 \\ \dot{r}_2 = \nu_4 r_1\cos(\varphi_1 - \varphi_2) \\ \dot{\varphi}_2 = \nu_4 r_1 \sin(\varphi_1 - \varphi_2) \end{cases}$$

We shall study the case when

$$\nu_3 = K_1 r_1, \quad \nu_4 = K_2 r_2, \quad K_j = conts., j = 1, 2$$

Hence, the solutions of the above equations are:

$$\begin{cases} r_1 = C_1 \\ \varphi_1 = C_1 t + C_2 \\ \varphi_2 = C_1 t + C_2 - \Psi(t) \\ r_2 = \exp(C_1\displaystyle\int_{t_0}^{t}\cos\Psi(t)dt)C_3 \end{cases}$$

where $\Psi(t) = \varphi_1 - \varphi_2$ is a solution of the equation

$$\int \frac{d\Psi}{1 - \frac{K_2}{K_1} \sin \Psi} = K_1 C_1 t + C_4$$

where $C_j, \quad j = 1, 2, 3, 4, \ K_1, K_2$ are arbitrary constants. It is easy to show that the given differential equations admits the following first integrals

$$r_1 = C_1, \quad r_2 (1 - \frac{K_2}{K_1} \sin(\varphi_1 - \varphi_2)) = C.$$

5. CARTESIAN APPROACH FOR NONHOLONOMIC SYSTEM WITH N DEGREE OF FREEDOM AND N-1 CONSTRAINTS .

We apply the obtained above results to solve the inverse problem in Dynamics [Ram].

One of the fundamental classical problems in celestial mechanics is to determine the potential-energy function U such that every curve from a given family of curves will be a possible trajectory of a particle moving under the action of potential forces F, admitting U; i. e. $F = gradU$.

The importance of this problem was already acknowledged Szebehely [Sze1]. Szebehely, indeed, affirms that in order to establish accurate physical descriptions and accurate constants, one needs to address the inverse problem of dynamics.

The first inverse problem in Celestial Mechanics was stated and solved by Newton (1687) [New] and concerns the determination of the potential field of force that ensures the planetary motion in accordance to the observed properties, namely to Kepler's laws.

Bertrand (1877) [Ber] proved that the expression for Newton's force of attraction can be obtained directly from the Kepler first law to within a constant multiplier.

Bertrand stated also a more general problem of determining a positional force, under which a particle describes a conic section under any initial conditions.

The ideas of Bertrand were developed by Dainelli [Dai], Suslov [Sus], Joukovski [Jou], .

Dainelli in [Dai] essentially states a more general problem of how to determine the most general field of force (the force being supposed to depend only on the position of the particle on which it acts) under which a given family of planar curves is a family of orbits of a particle.

The solution proposed by Dainelli is the following [Whi,Dain,Sad].

The most general field of force $\mathbf{F} = (F_x, \, F_y)$ capable of generating the family of planar orbits $f(x, y) = Const$ is the following:

$$\text{(5.1)} \qquad \begin{cases} F_x = -\lambda^2 \{f, \partial_y f\} - \lambda \{f, \, \lambda\} \partial_y f \\ F_y = \lambda^2 \{f, \partial_x f\} + \lambda \{f, \, \lambda\} \partial_x f. \end{cases}$$

where λ is an arbitrary function,

$$\{f,\,\lambda\} \equiv \partial_x f \partial_y \lambda - \partial_y f \partial_x \lambda$$

In [Sus], Suslov stated and solved a problem which was a further development of Bertrand's problem. He shows that, given a $(N-1)$-parametric family of orbits in the configuration space of a holonomic system with N degrees of freedom and a kinetic energy T, it is necessary to determine the potential field of force under which any trajectory of the family can be traced by the representative point of the system.

Suslov deduced the following system of linear partial differential equations with respect to the require potential function:

$$\begin{aligned}&\frac{\partial\theta}{\partial\triangle_k}\frac{\partial U}{\partial x^N} - \frac{\partial\theta}{\partial\triangle_N}\frac{\partial U}{\partial x^k} = \\ &\frac{U+h}{\theta}\Big(\frac{\partial\theta}{\partial\triangle_N}\frac{\partial\theta}{\partial x^k} - \frac{\partial\theta}{\partial\triangle_k}\frac{\partial\theta}{\partial x^N} + \\ &\sum_{m=1}^{N}\triangle^m\big(\frac{\partial\theta}{\partial\triangle_k}\frac{\partial^2\theta}{\partial\triangle_N\partial x^m} - \frac{\partial\theta}{\partial\triangle_N}\frac{\partial^2\theta}{\partial\triangle_k\partial x^m}\big)\Big)\end{aligned}$$

where θ, $\triangle^1$, $\triangle^2, ..., \triangle^N$ are functions:

$$\sum_{j=1}^{N}\frac{\partial f_\alpha}{\partial x^j}\triangle^j = 0, \quad \triangle_k = \sum_{j=1}^{N} G_{jk}(x)\triangle^j, \quad \alpha = 1,2,..,N-1,\ k = 1,2,..,N.$$

$$\theta = \frac{1}{2}\sum_{k,j=1}^{N} G_{kj}(x)\triangle^k\triangle^j \equiv \theta(x^1, x^2, .., x^N, \triangle_1,\, \triangle_2, ..., \triangle_N)$$

and proved that theses equations represented the necessary and sufficient conditions under which the equations of motion of the study mechanical system admits the given $N-1$ partial integrals.

Assuming that given trajectories admit a family of the orthogonal surfaces, Joukovski in [Jou] constructed the potential-energy functions in explicit forms for systems with two and three degrees of freedom.

The following theorem was enunciated by Joukovski in 1890, that

if $q = const$ is the equation of the family of curves on a surface, and $p = const$ denotes the family of curves orthogonal to these, then the curves $q = const$ can be freely described by a particle under the influence of forces derived from the potential-energy function

$$V = \Delta_1(p)\Big(g(p) + \int h(q)\frac{\partial}{\partial q}(\frac{1}{\Delta_1(p)})dq\Big)$$

where h and g are arbitrary functions, and Δ_1 denotes the first differential parameter

In the most general form the inverse problem in dynamics was studied in [Sad]. By applying the results presented in that work we propose the following new results:

1. Generalization the Dainelli problem for a mechanical system with N degree of freedom

2. New approach to solve the Suslov problem

3. Generalization of the Joukovski problem for a mechanical system with N degree of freedom

Generalized Dainelli's and Generalized Joukovski's problem

We introduce necessary notations and give a brief overview of the main results obtained in [Sad].

Definition 2.1 [Generalized Dainelli's problem].

Given a $N-1$ -parametric family of orbits

$$f_j(x) = C_j, \quad j = 1, 2, ..., N-1 \tag{5.2}$$

in the configuration space of a holonomic system with N degrees of freedom and kinetic energy

$$T = ||\dot{\mathbf{x}}||^2 = \frac{1}{2}\sum_{j,k=1}^{N} G_{kj}(x)\dot{x}^j\dot{x}^k.$$

The Generalized Dainelli problem is the problem of determining the most general field of force that depends only on the position of the system under which any trajectory of the family can be traced by a representative point of the system.

Proposition 5.1 (Solution of the Generalized Dainelli Problem).

Given a mechanical system $\mathcal{M}$ with configuration space X and a kinetic energy T. Then the most general field of force that depends only on the position of the system and is capable of generating the given orbits (5.2) is described by the equation (2.6), or what is the same,

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{x}^n} - \dfrac{\partial T}{\partial x^n} = \omega(\partial_n), \quad n = 1, 2, .., N \\ \omega = d\dfrac{\|\mathbf{v}\|^2}{2} + \imath_{\mathbf{v}} d\sigma, \end{cases} \tag{5.3}$$

Here $\imath_{\mathbf{v}}$ is the contraction along the vector field $\mathbf{v}$, $f_1, \dots, f_{N-1}$ are independent functions of class $C^r(\widetilde{X} \subseteq X)$, $r \geq 2$, $\mathbf{v}$ is the vector field (2.5), and σ is a 1-form associated with $\mathbf{v}$.

Definition [Suslov's problem]. Given a $(N-1)$-parametric family of orbits in the configuration space of a holonomic system with N degrees of freedom and kinetic energy T. The Suslov problem is the problem of determining the potential field of force under which any trajectory of the family can be traced by a representative point of the system.

Corollary 5.1 (Solution of the Suslov Problem).

$$\omega = dU(x) \Leftrightarrow \imath_{\mathbf{v}} d\sigma = -dh(f_1, f_2, ..., f_{N-1}) \tag{5.4}$$

If (5.4) holds then

$$U(x) = \frac{1}{2}\lambda^2(x)||\mathbf{v}||^2 + h(f_1, f_2, ..., f_{N-1}) \tag{5.5}$$

Definition [Generalised Joukovskis problem]. The generalized Joulovski problem is a particular case of the Suslov problem, which is obtained by assuming that the vector field (2.5) has the form :

$$\{f_1, f_2, \dots, f_{N-1}, * \quad \} = \nu(x)\nabla S, \tag{5.6}$$

where

$$\nabla S = \sum_{j=1}^{N} G^{jk}(x)\partial_j S \partial_k,$$

$G^{-1} = (G^{jk})$ is the inverse matrix of the Riemann metric G and ν is a certain function.
The stated problem coincides with Joukovski's problem when $N = 3$. [Whit, Jou]
Corollary 5.2 (Solution of the Joukovski Problem)

$$\omega = dU(x) \Leftrightarrow \imath_{\Gamma\nabla S} d(\Gamma dS) = -dh(f_1, f_2, ..., f_{N-1}), \quad \Gamma \equiv \lambda\nu \tag{5.7}$$

Hence, in view of (5.7) we obtain

$$U(x) = \frac{1}{2}\Gamma^2(x)||\nabla S||^2 + h(f_1, f_2, ..., f_{N-1})$$

Corollary 5.3
If

$$\Gamma = \Gamma(S)$$

then

$$U(x) = \Gamma^2(S)||\nabla S||^2 + h_0$$

Corollary 5.4 The field of force (5.3) takes for a particle in R^2 the following forms:

$$\omega = d\frac{1}{2}\lambda^2((\partial_x f)^2 + (\partial_y f)^2) + \lambda\big(\partial_x(\lambda f_x) + \partial_y(\lambda f_y)\big)df \tag{5.8}$$

It is possible to show that (5.8) coincide with (1.1)[Ram].
Corollary 5.5 The field of force (5.3) for a particle in R^2 is potential iff

$$\lambda\big(\partial_x(\lambda f_x) + \partial_y(\lambda f_y)\big)df = -dh(f)$$

and respectively in R^3, iff

$$\lambda\imath_{curl\,\mathbf{v}}(df_1 \wedge f_2) = -dh(f_1, f_2)$$

where $\mathbf{v} = \lambda(x, y, z)[\,grad\, f_1 \times \, grad\, f_2]$, $[\times]$ is the vector product on R^3.
Clearly, if $\mathbf{v}$ is a Kummer vector field the $dh = 0$.

In 1974 Szebehely [Sze2] obtained a linear first-order partial differential equation for the potential function U which gives rise to a one-parameter family of planar orbits with a given total energy h. This result originated many works on the inverse problems (see for instance [Boz]). The equation of Szebehely was generalized to a two-parameter family of three-dimensional orbits by Bozis (1983) .

We show that the results, presented in those works, can be obtained from the solutions of the Suslov problem.

By applying the above results we prove the following result

Corollary 5.6 (Solution of the Bertrand Problem)

The potential-energy function U capable of generating a one-parameter family of conics with eccentricity $b \neq 0$

$$r(1 + b\cos\theta) = Const$$

is the function

$$U = a_{-1}(H_1(\cos\theta) - K_1 \log r(1 + b\cos\theta)) + \sum_{j\in Z\setminus\{-1\}} a_j r^{j+1}(H_j(\cos\theta) - \frac{1 + b\cos\theta}{j+1})$$

where $a_j\, K_1 \quad j \in Z$, are real constants and H_j, $j \in Z$ are solutions of the Heun equations with singularities at the points

$$0,\, 1\, , \frac{1+b}{b}, \infty$$

and with the exponents

$$(0, \frac{j+3+b(j+1)}{2b});\;\; (0, j - \frac{j+3+b(j+1)}{2b});\; (0, j+1); (-1-j, 1-j)$$

respectively.

The solution of this problem for the case when $b = 0$ it is easy to obtain.

We prove the following proposition which represented an extension of the Joukovski theorem for a mechanical system with N degree of freedom.

Proposition 5.2

If

$$x^j = C_j = const, \quad j = 1, 2, .., N-1 \tag{5.9}$$

are the equations of the $N-1$ parametric family of curves on X, and $x^N = const$ denotes the family of curves orthogonal to these, then the curves (5.9)can be freely described by a representative particle under the influence of forces derived from the potential-energy function

$$U = \frac{1}{G_{NN}(x^1, x^2, .., x^N)}\Big(g(x^N) + \sum_{j=1}^{N-1}\int h(x^1, x^2, .., x^{N-1})\frac{\partial G_{NN}(x^1, x^2, ..x^N)}{\partial x^j}dx^j\Big)$$

where h and g are arbitrary functions.

Clearly, for $N = 2$ we exactly obtain the Joukovski result given in the introduction.

Definition [Stäckel system]. The Stäckel system is the triplet

$$\mathcal{M} =< X,\, T = \frac{1}{2}\sum_{k=1}^{N} \frac{\dot{x}_k^2}{A^k(x)},\, \omega = dU(x)\ >,$$

where $A^1, ..., A^N, U$ are functions that :

$$A^k(x) = \frac{1}{\Delta}\frac{\partial \Delta}{\partial \varphi_{k1}(x^k)}, \tag{5.10}$$

$$U(x) = \sum_{k=1}^{N} \Psi_k(x^k) A^k, \tag{5.11}$$

$$\Delta = \det \begin{pmatrix} d\varphi_1(\partial_1) & \dots & d\varphi_1(\partial_N) \\ \vdots & \vdots & \vdots \\ d\varphi_N(\partial_1) & \dots & d\varphi_N(\partial_N) \end{pmatrix} = d\varphi_1 \wedge ... \wedge d\varphi_N(\partial_1, ..., \partial_N),$$

$$d\varphi_\alpha = \sum_{k=1}^{N} \varphi_{k\alpha}(x^k) dx^k,$$

$\varphi_{k\alpha}$, Ψ_k are arbitrary functions, $k = 1, ..., N$, $\alpha = 2, ..., N$.

The trajectories of the Stäckel system are [Char]:

$$f_{\mu-1}(x) \equiv \sum_{k=1}^{n} \int \frac{\varphi_{k\mu}(x^k)}{\sqrt{K_k(x^k)}} dx^k = c_\mu, \quad \mu = 2, ..., N, \tag{5.12}$$

where $K_k(x^k) = 2\Psi_k(x^k) + 2\sum_{j=1}^{N} \alpha_j \varphi_{kj}(x^k)$, α_j, $k = 1, 2, ..N$ are constants.

Corollary 5.7

The system (2.5)+(5.11) can be rewritten as follow

$$\begin{cases} \dot{\mathbf{x}} = \dfrac{\lambda}{\prod_{n=1}^{N} K_n(x^n)} \det \begin{pmatrix} d\varphi_1(\partial_1) & \dots & d\varphi_1(\partial_N) \\ \vdots & & \vdots \\ d\varphi_{N-1}(\partial_1) & \dots & d\varphi_{N-1}(\partial_N) \\ \partial_1 & \dots & \partial_N \end{pmatrix} \\ \Upsilon = \Delta \neq 0, \quad df_N = d\varphi_N \end{cases}$$

Clearly, if

$$\lambda = \frac{\prod_{n=1}^{N} K_n(x^n)}{\Delta}$$

then (5.10) and (5.11) can be rewritten as follows

$$A^n(x) = \mathbf{v}(x^n)$$
$$U(x) = d\psi(\mathbf{v})$$

We define the *inverse Stäckel problem* as follows [Sad]

Definition [Inverse Stäckel problem]. Let $\mathcal{M}$ be a mechanical system with configuration space X and kinetic energy

$$T = \frac{1}{2}\sum_{k=1}^{N}\frac{\dot{x}_k^2}{A^k(x)},$$

where $A^1, ..., A^N$ are function determine by formulae (5.10)

The problem of constructing the potential field of force

$$\omega = -dU(x,y),$$

which is capable of generating orbits (5.12)is called the inverse Stäckel problem.

We obtain that the solution to the inverse Stäckel problem is the function U :

$$U(x) = \Gamma^2(S)(\sum_{k=1}^{N}\Psi_k(x^k)A^k + \alpha) - h_0$$

where

$$S = \sum_{n=1}^{N}\int\sqrt{K_n(x^n)}dx^n$$

In particular if $\Gamma = 1$, $\alpha = h_0$, then $U = \sum_{k=1}^{N}\Psi_k(x^k)A^k$.

REFERENCES

[Bat] Bates, L. y Sniatycki, J., Nonholonomic reduction, Reports on Math.Physics, vol.32, 1993.

[Ber] Bertrand, M.I., Sur la posibilité de déduire d'une seule de lois de Kepler le principe de l'attraction. Comtes rendues, **9**. (1877)

[Bloch] Bloch A.M., Nonholonomic Mechanics and Control, Springer-Verlag, 2003.

[Boz] Bozis, G. , The inverse problem of dynamics: basic facts, Inverse Problems **11**, 687-708, 1995. Mech., **38**, 357, 1986

[Charlier] Charlier C. L.:1966, Celestial Mechanics (Die Mechanik Des Himmels) Ed." Nauka", Moscow (in Russian).

[Dai] Dainelli, U., Sul movimento per una linea qualunque, Giorn. Mat., **18**, (1880)

[Dub] Duboshin, G.H., Celest. Mech., Ed. Nauka, Moscow, 1968 (in Russian).

[Erm] Ermakov, V.P., Determination of the potential fucntion from given partial integrals, Math. Sbornik 15, serie 4, 1881 (in Russian)

[Gal] Galiullin, A.S., Inverse Problems of dynamics, Mir Publishers, Moscow, 1984

[Gantmacher] Gantmacher, F.P., Lektsii po analiticheskoi dinamike Ed. "Nauka", Moscú, 1966 (in Russian).

[Jou] Joukovski, N.E. Postroenye potencialnaia funksia po zadannie cemiestvo trayectories. "Sobranye sochinyeni",T.1, ed Gostexizdat, 1948, p.227-242.

[Kharl] E.I. Kharlamov-Zabelina, Rapid rotation of a rigid body about a fixed point under the presence of a nonholonomic constraints. Vestnik Moskovsk. Univ., Ser. Math. Mekh., Astron., Fiz.Khim., V.6, 1957, pp.25-34 (in Russian).

[Kle] Klein, J. Espaces variationnels et mecànique, Ann Inst. Fourier, 12, 1962. p 1-124.

[Koz1] Kozlov, V.V., Obshaia teria vixrei, Udmurski Universitiet, 1998 (in Russian).

[Koz2] Kozlov, V.V., Symmetries, Topology and Resonances in Hamiltonian Mechanics. Springer-Verlag. Berlin, 1996.

[Krat] Kratzer, A. and Franz W., Transzendente Funktionen, Leipzig 1960.

[NF] Neimark,Yu. and Fufaev, N.A., Dynamics of nonholonomic Systems. Amer. Math. Soc. Translation 33, 1972.

[Mac] Maciejewski A.and Przybylska M., Nonintegrability of the Suslov problem, Journal of Mathematical Physics, vol.45, number 3, p.1065-1078 (2004).

[Poliajov] Poliajov, N.N. et all, Mekanica teorika, Ed.LGU. 1985 (in Russian).

[Ram] Ramirez, R. and Sadovskaia N.,Cartesian approach for Nonholonomic systems, J. of Math. Sciences, vol. 128, Number 2 (2005), 2812-2817.

[Ros] Rosenberg, R., Analytical Dynamics, Ed. Heinemann, New York, 1977.

[Sad] Sadovskaia, N. Inverse problem in theory of ordinary differential equations. Thesis Ph. Dr., Univ. Politécnica de Cataluña , 2002 (in Spanish)

[Sus] Suslov, G.K. Opredilienyie silovoi funkcyi po zadanniem shasnim integralam, Kiev, 1890.

[Sze1] Szebehely, V. Open problems on the eve of the next millenium. Celest. Mech., 65, 205, 1997.

[Szeb2] Szebehely, V.:1974, On the determination of the potential, E. Proverbio, Proc. Int. Mtg. Rotation of the Earth, Bologna.

[Whi] Whittaker, E. T.:1959, A treatise on the Analytic Dynamics of Particles and Rigid Bodies.-Cambridge: Cambridge University, Press.