IS IT POSSIBLE TO SOLVE THE DYNAMICS PROBLEM INSIDE THE CONFIGURATION SPACE?

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Abstract

In the history of mechanics, there have been two points of view for studying mechanical systems: The Newtonian and the Cartesian.

A main achievement of Newton was perceiving that the dynamics of the real system are prescribed by second-order differential equations. To reduce the equations of motion to the investigation of dynamic system it is necessary to double the dimension of the position space and to introduce the auxiliary phase space.However, we are interested not in the phase trajectories themselves but in their projection on the configuration space.

According the Decartes point of view, the motion of mechanical systems is described by the first-order differential equations in the N dimensional configuration space $\mathcal{Q}$:

$$\dot{x} = \mathbf{v}(x,t), \quad x \in \mathcal{Q}.$$

Decartes gave no principles for constructing the field $\mathbf{v}$ (the Decartes vector field).

V.V. Kozlov in the monograph *Dynamical systems X* affirms that solving dynamics problems is possible inside the configuration space.

In this paper we develop the Cartesian approach for mechanical systems with constraints which are linear with respect to velocity. We show that the Decartes vector field in this case admits the following representation

$$\mathbf{v}(\mathbf{x}) = \sum_{j=M+1}^{N} \lambda_j(x) X_j$$

where $X_{M+1}, ...X_N$ are characteristic elements of the independents 1-forms Ω_1, $\Omega_2, .., \Omega_M$, the functions λ_{M+1}, $\lambda_{M+2}, ..., \lambda_N$ are such that the 1-form σ associated to the Decartes vector field admits the representation

$$\imath_{\mathbf{v}} d\sigma = \sum_{j=1}^{M} \Lambda_j(x)\Omega_j$$

where by $\imath_{\mathbf{v}}$ we denote the contraction along $\mathbf{v}$.

The obtained results are illustrated into the study of the particular case when the the smallest Lie algebra of vector fields on $\mathcal{Q}$ that contains the vector field $X_{M+1}, ..., X_N$ is finite dimensional.

Key words: Non-holonomic systems, Cartesian approach, Newtonian approach, constraint, differential equation, Lagrangian systems, Suslov's problem, Veselov's problem, rattleback .

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1. Introduction

In "Philosophiae Naturalis Principia Mathematica" (1687), Newton considers that movements of celestial bodies can be described by differential equations of the second order. To determine their trajectory, it is necessary to give the initial position and velocity.

Descartes in 1644 proposed that the behavior of the celestial bodies be studied from another point of view. These ideas were stated in "Principia Philosophiae" (1644) and in "Discours de la métode" (1637). According to Descarte the understanding of cosmology starts from acceptance of the initial chaos, whose moving elements are ordered according to certain fixed laws and form the Cosmo. He consider that the Universe is filled with a tenuous fluid matter (ether), which is constantly in a vortex motion. This motion moves the largest particle of matter of the vortex axis, and they subsequently form planets. Then, according to what Descartes wrote in his "Treatise on Light", "the material of the Heaven must be rotate the planets not only about the Sun but also about their own centers...and this will hence form several small Heavens rotating in the same direction as the great Heaven."

Newton gave a simpler, but stronger, argument against Descarte's theory. If the Descarte's ideas is correct, bodies are carried by the ether, and the equations of motion are consequently of first order: the velocity of a particle depend only on its position. However, Newton noted that some of the observed comets move in a direction opposite to that of all the planets [Koz1].

In the modern scientific literature the study of the Descarte ideas we can find in the monographic of V.V. Kozlov in which the author said "In the present book, one more attempt is made to rehabilitate Descarte's vortex theory..." . In this books, Kozlov affirms "solving dynamics problem is possible inside the configuration space".

As we observe , the equation of motion in the Descartes theory must be of the first order

$$\dot{\mathbf{x}} = \mathbf{v}(x, t) \tag{1.1}$$

Hence, to determine the trajectory from Descartes's point of view it is necessary to give only the initial position. Descarte gave no principles for constructing the field $\mathbf{v}$ for different mechanical systems.

Definition

The vector field (1.1) we shall call the Decartes vector field.

The aim of the present paper is to develop the Descarte ideas for mechanical systems with constraints which are linear with respect to the velocity.

2. CONSTRUCTION THE CARTESIAN VECTOR FIELD FOR NON-HOLONOMIC SYSTEM

Firstly we shall introduce the following notation and concepts.

Let $\mathcal{Q}$ be a smooth manifold of the dimension N with local coordinates $x = (x^1, ..., x^N)$ and equipped by the Riemann metric $G = (G_{kj}(x))$.

By $\xi(\mathcal{Q})$, $\Lambda(\mathcal{Q})$, ∇ we denote respectively the Lie algebra of vector fields on $\mathcal{Q}$ and the algebra of the 1-form on $\mathcal{Q}$, and the connection:

$$\begin{aligned} \nabla : \xi(\mathcal{Q}) \times \xi(\mathcal{Q}) &\longmapsto \xi(\mathcal{Q}) \\ (u, v) &\longmapsto \nabla_u v \end{aligned}$$

which is R lineal with respect to v and C^∞ lineal with respect to u and is compatible with metric G, i.e., $\nabla_u G(v, w) = 0, \quad \forall u, v, w \in \xi(\mathcal{Q})$.

Let $\mathbf{v} \in \xi(\mathcal{Q})$ be a vector field:

$$\mathbf{v} = \frac{det}{\Upsilon}\begin{pmatrix} \Omega_1(\partial_1) & \Omega_1(\partial_2) & \dots & \Omega_1(\partial_N) & \lambda_1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_M(\partial_1) & \Omega_M(\partial_2) & \dots & \Omega_M(\partial_N) & \lambda_2 \\ \Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \dots & \Omega_{M+1}(\partial_N) & \lambda_{M+1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_N(\partial_1) & \Omega_N(\partial_2) & \dots & \Omega_N(\partial_N) & \lambda_N \\ \partial_1 & \partial_2 & \dots & \partial_N & 0 \end{pmatrix}, \tag{2.1}$$

where $\Upsilon \equiv \Omega_1 \wedge \Omega_2 ... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N)$, $\partial_k = \frac{\partial}{\partial x^k}$, we shall consider that $\Omega_1, \Omega_2, ...\Omega_N$ are 1-forms on $\mathcal{Q}$:

$$\Omega_j = \sum_{j=1}^{N} b_{jk}(x) dx^k + w dx^j, \quad j = 1, 2, .., N, \quad w \in \mathrm{C}$$

Furthermore, we assume that they are pointwise independent i.e. $\Upsilon \neq 0$.

The functions $\lambda_j, \quad j = M+1, ..., N$ are arbitrary functions on $\mathcal{Q}$:

$$\Omega_j(\mathbf{v}) = \lambda_j, \quad j = 1, 2, .., N$$

The vector $\mathbf{v}(x) = (v^1(x), ..., v^N(x))^T$ can be represented as follows

$$\mathbf{v}(x) = \mathcal{M}^{-1}\lambda,$$

where $\mathcal{M} = \Big(\Omega_j(\partial_k)_{j,k=1,..N}\Big), \quad \lambda = -(\lambda_1, .., \lambda_M, \lambda_{M+1}, ..\lambda_N)$. or, what is the same,

$$\mathbf{v} = \sum_{j=1}^{N} \lambda_j X_j \tag{2.1}$$

where X_j, $j = 1, ..., N$ constitute a set of independent vector fields on $\mathcal{Q}$

The differential equations generated by the vector field $\mathbf{v}$ are the following

$$\dot{\mathbf{x}} = \mathcal{M}^{-1}\lambda = \sum_{j=1}^{N} \lambda_j X_j \tag{2.2}$$

Let σ be the 1-form associated with the vector field $\mathbf{v}$, i.e.,

$$\sigma = (\mathbf{v}(x), dx) \equiv \sum_{j,k=1}^{N} G_{jk}(x) v^j(x) dx^k \equiv \sum_{k=1}^{N} p_k dx^k$$

then the 2-form $d\sigma$:

$$d\sigma = \frac{1}{2} \sum_{j,k=1}^{N} a_{jk}(x) \Omega_j \wedge \Omega_k,$$

where $A = (a_{jk})$ is a matrix such that

$$a_{jk} = (-1)^{j+k-1} \frac{1}{\Upsilon} d\sigma \wedge \Omega_1 \wedge ... \wedge \widehat{\Omega}_k .... \wedge \widehat{\Omega}_j .... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N)$$

$\widehat{\Omega}_j$, $\widehat{\Omega}_k$ means that these elements are omitted.

It is clear that the contraction of $d\sigma$ along $\mathbf{v}$ is

$$\iota_{\mathbf{v}} d\sigma = \sum_{j=1}^{N} \Lambda_j \Omega_j,$$

where

$$\Lambda \equiv col(\Lambda_1, \Lambda_2, ..., \Lambda_N) = A^T \lambda.$$

where $A = (a_{jk}(x))$

Proposition 2.1 *The differential equations*

$$\dot{\mathbf{x}} = \mathbf{v}(x), \quad x \in \mathcal{Q}$$

are invariant relationship of the Lagrangian equations with Lagrangian function

$$L = \frac{1}{2} ||\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})||^2 \equiv \frac{1}{2} \sum_{j,k=1}^{N} G_{kj}(x)(\dot{x}^j - v^j(x))(\dot{x}^k - v^k(x))$$

In fact, by derivation we deduce that $\nabla_{\dot{\mathbf{x}}}(\dot{\mathbf{x}} - \mathbf{v}(x)) = 0$, or,

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{\mathbf{x}}} L) = 0,$$

which are equivalent to Lagrangian equations with the Lagrangian function L_0 given above. It is easy to show that these equations admits the representation

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{x}^j} T) = \omega(\partial_j) + \nabla_{\dot{\mathbf{x}}-\mathbf{v}(\mathbf{x})} p_j \tag{2.3}$$

where

$$T = \frac{1}{2}||\dot{\mathbf{x}}||^2, \quad p_j = \sigma(\partial_j)$$
$$\omega = d\frac{||\mathbf{v}||^2}{2} + \iota_{\mathbf{v}} d\sigma,$$

The aim of the following assertions is to determine the Cartessian approach in the next two cases:

CASE I

In this case we shall consider that the functions $\lambda_1, \lambda_2, ..., \lambda_N$ are such that

$$\lambda_j = 0, \quad j = 1, 2, ..M$$

and the 1-form $\Omega_1, \Omega_2, ..., \Omega_M$ are given 1-forms for which the vector field $X_{M+1}, X_{M+2}, .., X_N$ constitute a maximal set of independent vector fields on $\mathcal{Q}$ satisfying the constraints in the sense that the components of X_j satisfy the equations

$$\Omega_j(X_k) = 0, \quad j = 1, 2, .., M, \quad k = M+1, ..., N,$$

i.e., X_j are characteristic elements of the given 1-forms, and $\Omega_{M+1}, ..., \Omega_N$ are arbitrary 1-forms such that $\Upsilon \neq 0$

CASE II

Now we determine the functions $\lambda_1, \lambda_2, ..., \lambda_N$ and the 1-form as follows

$$\begin{cases} \lambda_j = a_j(x), \quad j = 1, 2, ..., N \\ \Omega_j = \displaystyle\sum_{k=1}^{N} (da_j(\partial_k) + da_k(\partial_j))dx^k + 2wdx^j \end{cases}$$

where w is an arbitrary parameter which we determine in such away that

$$\begin{cases} \Omega(\mathbf{v}) = 0 \\ \Omega = \displaystyle\sum_{j=1}^{N} a_k(x)dx^k \end{cases}$$

THE CARTESSIAN APPROACH FOR THE CASE I

We shall study the case I.

The differential equations which describe the behavior of such mechanical systems under the restrictions that

$$\imath_{\mathbf{v}} d\sigma = \sum_{j=1}^{M} \Lambda_j(x)\Omega_j$$

can be represented as follows

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{x}^k} T) = \frac{\partial \frac{1}{2}||\mathbf{v}||^2}{\partial x^k} + \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_k), \quad k = 1, 2, ..N, \tag{2.4}$$

and can be interpreted as the equations of motion of non-holonomic mechanical systems with an active potential field of force with potential U :

$$U = \frac{1}{2}||\mathbf{v}(x)||^2 + U_0, \quad U_0 = const.$$

and with the reactive forces with the components

$$(\sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_1), \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_2), ..., \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_N)),$$

generated by the constraints

$$\Omega_j(\dot{\mathbf{x}}) \equiv \sum_{k=1}^{N} \Omega_j(\partial_k)\dot{x}^k = 0, \quad j = 1, 2, ..N.$$

The differential equations generated by the vector field under the indicated restrictions can be represented as follows

$$\dot{\mathbf{x}} = \mathcal{M}^{-1}\lambda = \sum_{j=M+1}^{N} \lambda_j X_j \tag{2.5}$$

where λ_j, $j = M + 1, M + 2, ..., N$ and the arbitrary 1-form Ω_j, $j = M + 1, .., N$ are such that

$$\begin{cases} \Lambda_j(x) = 0, \quad j = M + 1, ..., N \\ \Upsilon \equiv \Omega_1 \wedge \Omega_2 ... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N) \neq 0 \end{cases} \tag{2.6}$$

Definition

The studying of the behavior of the non-holonomic systems by using the equations (2.5), (2.6) or (2.4) we called *Cartesian and Lagrangian approach* respectively and by applying the equations deduced from the D'Alembert-Lagrange Principle we called the *Classical approach.*

Corollary 2.2

If

$$\begin{cases} M = N-1 \\ \Omega_j = df_j(x), \quad j = 1,2,...,N-1 \\ \Omega_N = df_N \end{cases}$$

Then the equations (2.5), (2.6) and (2.4) take the form respectively

$$\begin{cases} \dot{\mathbf{x}} = \lambda_N \det \begin{pmatrix} df_1(\partial_1) & \dots & df_1(\partial_N) \\ \vdots & & \vdots \\ df_{N-1}(\partial_1) & \dots & df_{N-1}(\partial_N) \\ \partial_1 & \dots & \partial_N \end{pmatrix} \\ \Upsilon = df_1 \wedge df_2 .... \wedge df_N(\partial_1, \partial_2, ..., \partial_N) \neq 0 \end{cases}$$

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{x}^k} T) = \frac{\partial \frac{1}{2}||\mathbf{v}||^2}{\partial x^k} + \sum_{j=1}^{N-1} \lambda_N a_{Nj}(x) df_j(\partial_k).$$

where λ_N is an arbitrary function.

The Lagrangian approach for this particular case we applied to solve the inverse problem in dynamics [Ram1, Ram2, Ram3, Sad]

With respect to the proposed us approaches we have the following conjecture.

Conjecture

The Cartesian and Lagrangian approach are equivalent.

This conjecture supported the following facts. First, the solutions of (2.5),(2.6) are solutions of (2.4) in view of proposition 2.1. Second, the solutions of the equations (2.4) depend on the $2N - M$ initial conditions. The solutions of (2.5) depend on N initial conditions and $N - M$ functions which are solutions of the linear partial differential equations of first order (2.6).

To illustrate this conjecture we study the following example.

A NON-HOLONOMICALLY CONSTRAINED PARTICLE IN R^3.

Consider a particle with the kinetic energy

$$T = \frac{1}{2}((\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2)$$

and non-holonomic constraints

$$\Omega_1(\dot{\mathbf{x}}) \equiv \dot{x} + a(z)\dot{y} = 0$$

This instructive academic example, in the particular case when $a(z) = z$, due to Rosenberg [Ros]. This example was also used to illustrate the theory in Bates and Sniatycki [Bates]. The Cartesian approach in this case produces the following vector field $\mathbf{v}$:

(2.7) $$\mathbf{v} = \lambda_2(a(z)\partial_x - \partial_y) - \lambda_3\partial_z = \lambda_2 X_2 + \lambda_3 X_3$$

and condition (2.6) for this case takes the form

$$\lambda_2\Omega_1(rot\mathbf{v}) = \frac{1}{2}\partial_z((1+a^2)\lambda_2^2) + (a\partial_x\lambda_3 - \partial_y\lambda_3)\lambda_2 = 0. \tag{2.8}$$

The vector field X_2 and X_3 are such that

$$\begin{cases} X_2 = a(z)\partial_x - \partial_y \\ X_3 = \partial_z \\ [X_3, X_2] = \partial_z a(z)\partial_x \end{cases} \tag{2.9}$$

We shall study the case when in (2.8)

$$\lambda_2 = \frac{A}{\sqrt{a^2+1}}, \qquad \lambda_3 = b_2(z),$$

for A an arbitrary constant and b_2 an arbitrary function.
The equations generated by the vector field $\mathbf{v}$ in this case are

$$\begin{cases} \dot{x} = \dfrac{a(z)A}{\sqrt{1+a^2(z)}} \\ \dot{y} = -\dfrac{A}{\sqrt{1+a^2(z)}} \\ \dot{z} = -b_2(z) \end{cases}$$

Hence the all trajectories of these equations are the following

$$\begin{cases} x = x_0 - A\displaystyle\int_{z_0}^{z} \frac{a(z)dz}{b_2(z)\sqrt{1+a^2(z)}} \\ y = y_0 - A\displaystyle\int_{z_0}^{z} \frac{dz}{b_2(z)\sqrt{1+a^2(z)}} \\ t = t_0 - \displaystyle\int_{0}^{z} \frac{dz}{b_2(z)} \end{cases} \tag{2.10}$$

The equation (2.4) may be rewritten as

$$\begin{cases} \ddot{x} = -b(z)\partial_z(\dfrac{Aa(z)}{\sqrt{a^2(z)+1}}) \\ \ddot{y} = b(z)\partial_z(\dfrac{A}{\sqrt{a^2(z)+1}}) \\ \ddot{z} = \partial_z\dfrac{b_2^2(z)}{2} \end{cases}$$

Corollary 2.3
All the trajectories of the equation of motion of the constrained Lagrangian system

$$< \mathrm{R}^3,\, L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(z),\, \{\dot{x} + a(z)\dot{y} = 0\} >$$

can be obtained from (2.10) [Sad].

In fact, the equations of motion obtained from the D'Alembert-Lagrange Principle are

$$\begin{cases} \ddot{x} = \mu \\ \ddot{y} = a(z)\mu \\ \ddot{z} = \partial_z U(z) \\ \dot{x} + a(z)\dot{y} = 0 \end{cases}$$

Therefore,

$$\frac{d}{dt}(\dot{y} - a(z)\dot{x}) = -\frac{da(z)}{dz}\dot{z}\dot{x}$$

which, in view of the constraints, is equivalent to

$$\frac{\frac{d}{dt}(\dot{y}(1 + a^2(z))}{\dot{y}(1 + a^2(z))} = \frac{-a(z)\frac{da(z)}{dz}\dot{z}}{1 + a^2(z)},$$

hence,

$$\dot{y} = \frac{C}{\sqrt{1 + a^2(z)}}$$

where C is an arbitrary constant.

On the other hand from the equation

$$\ddot{z} = \partial_z U(z)$$

we easily obtain $\dot{z} = \pm\sqrt{2(U(z) + h)}$.

Finally by considering the constraints we deduce the system of the first order ordinary differential equations

$$\begin{cases} \dot{x} = -\frac{a(z)C}{\sqrt{1 + a^2(z)}} \\ \dot{y} = \frac{C}{\sqrt{1 + a^2(z)}} \\ \dot{z} = \pm\sqrt{2(U(z) + h)} \end{cases}$$

which coincide with the system obtained from the Cartesian approach if

$$C = -A, \quad b(z) = \mp\sqrt{2(U(z) + h)}.$$

In this example the Cartesian, the lagrangian and Classical approach coincide [Sad].

With respect to the arbitrary 1-form we posed the following problem.

Problem

Determine the 1-form $\Omega_{M+1}, ...\Omega_N$ from the condition that the smallest Lie algebra of vector fields on $\mathcal{Q}$ that contains the vector field $X_{M+1}, ..., X_N$ is finite dimensional.

If we assume that

$$X_1, X_2, .., X_M, X_{M+1}, .., X_N, ..., X_S$$

is a basis of this Lie algebra then

$$[X_j,\, X_k] = \sum_{m=1}^{S} C_{jk}^m X_m, \quad j, k = 1, 2, ..S$$

where $X_j \in \xi(\mathcal{Q})$, $j = 1, 2, ..S$ and $[X, Y]$ is the Lie brackets of vector field X and Y, and C_{jk}^m are the structure constants.

When the algebra is three dimensional then from the Bianchi representation we obtain:

$$\begin{cases} [X_1, X_2] = aX_2 + b_3 X_3 \\ [X_2, X_3] = b_1 X_1 \\ [X_3, X_1] = b_2 X_2 - aX_3 \end{cases} \tag{2.11}$$

where a, b_1, b_2, b_3 are certain constants

In the example 1 we obtain a finite Lie algebra if in (2.9) the function a is such that

$$1)\, a(z) = z, \quad 2)\, a(z) = \exp z, \quad 3)\, a(z) = \cos z,$$

A brief calculation shows that for the first case

$$\begin{cases} X_1 = \partial_z, \quad X_2 = z\partial_x - \partial_y, \quad X_3 = \partial_x \\ [X_1, X_2] = X_3, \quad [X_1, X_3] = 0, \quad [X_2, X_3] = 0 \end{cases}$$

which correspond to the Heisenberg algebra [Bloch].

For the second case we have

$$\begin{cases} X_1 = \partial_z, \quad X_2 = \exp z\partial_x - \partial_y, \quad X_3 = \partial_y \\ [X_1, X_2] = X_3 + X_1, \quad [X_1, X_3] = 0, \quad [X_2, X_3] = 0 \end{cases}$$

For the case when $a(z) = \cos z$ by introducing the vector field X_1, X_2, X_3, X_4 :

$$\begin{cases} X_1 = -\sin z\partial_x - \partial_y \\ X_2 = \cos z\partial_x - \partial_y \\ X_3 = \partial_z \\ X_4 = \partial_y \end{cases}$$

we deduce the four dimensional Lie algebra:

$$\begin{cases} [X_2, X_3] = X_3 + X_4 \\ [X_3, X_1] = -X_2 - X_4 \\ [X_2, X_1] = 0, \quad [X_2, X_4] = 0, \quad [X_3, X_4] = 0, \quad [X_1, X_4] = 0 \end{cases}$$

The all examples which we give below are such that the smallest Lie algebra of vector fields on $\mathcal{Q}$ that contains the vector fields $X_1, ..., X_N$ is finite dimensional.

3. CARTESIAN APPROACH FOR NON-HOLONOMIC SYSTEM WITH THREE DEGREE OF FREEDOM AND ONE CONSTRAINTS .

The case when $dim\mathcal{Q} = 3$ and $M = 1$ is of specific interest. We consider a natural mechanical system with configuration space $\mathcal{Q}$ and kinetic energy

$$T = \frac{1}{2} \sum_{k,j=1}^{3} G_{kj}(x)\dot{x}^j \dot{x}^k.$$

Obviously, in this case the 1-form $\iota_{\mathbf{v}} d\sigma$ can be represented as follow

$$\iota_{\mathbf{v}} d\sigma = \Lambda_1 \Omega_1 + \Lambda_2 \Omega_2 + \Lambda_3 \Omega_3,$$

where $\Lambda_j,\ j = 1, 2, 3$:

$$\begin{cases} \Lambda_1 = \Omega_2 \wedge \Omega_3(\mathbf{v}, rot\mathbf{v}) \\ \Lambda_2 = \lambda_3 \Omega_1(rot\mathbf{v}) \\ \Lambda_3 = -\lambda_2 \Omega_1(rot\mathbf{v}) \\ rot\mathbf{v} = \dfrac{1}{\sqrt{detG}} \big((\partial_y p_3 - \partial_z p_2)\partial_x + (\partial_z p_1 - \partial_x p_3)\partial_y + (\partial_x p_2 - \partial_y p_1)\partial_z, \big) \\ p_k = \displaystyle\sum_{j=1}^{3} G_{kj} v^j,\ k = 1, 2, 3 \end{cases} .$$

The system (2.2), (2.6) take the form respectively

(3.1) $$\dot{\mathbf{x}} = [\Omega_1 \times, \lambda_2 \Omega_3 - \lambda_3 \Omega_2] \equiv \lambda_2 X_2 + \lambda_3 X_3,$$

(3.2) $$\begin{cases} \Upsilon \neq 0 \\ \Omega_1(rot\mathbf{v}) = 0 \end{cases},$$

where $\Omega_j(x) = (\Omega_j(\partial_x), \Omega_j(\partial_x)\Omega_j(\partial_x)),\ j = 1, 2, 3;$ $[\ \ \times, \ \ ]$ is the vector product on R^3.

Definition

The vector field $\mathbf{v}$ we call the Kummer vector field if

$$[\mathbf{v} \times rot\mathbf{v}] = \mathbf{0}$$

It is easy to show that the equations of motion (2.4) for $N = 3$ under the condition that $\mathbf{v}$ is a Kummer a vector field can be represented in Lagrangian form.

The aim of the following assertions is to illustrate the above results in the concrete examples.

HEAVY RIGID BODY IN THE SUSLOV CASE

In this section we study one classical problem of non-holonomic dynamics formulated by Suslov [Koz2]. In this problem we consider the rotational motion of a rigid body around a fixed point and subject to the non-holonomic constraints $(\mathbf{a}, \omega) = 0$ where ω is a body angular velocity and $\mathbf{a}$ is a constant vector. Suppose the body rotates in an force field with potential $U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers we write the equations of motion in the form

$$\begin{cases} I\dot{\omega} = [I\omega \times \omega] + [\gamma \times \dfrac{\partial U}{\partial \gamma}] + \mu\mathbf{a} \\ \dot{\gamma} = [\gamma \times \omega] \\ (\mathbf{a}, \omega) = 0 \end{cases}$$

Where

$$I = diag(I_1, I_2, I_3),$$
$$\gamma = (\gamma_1 = \sin z \sin x, \quad \gamma_2 = \sin z \cos x, \quad \gamma_3 = \cos z)$$

I_1, I_2, I_3 are the inertial moment of the body.

If we assume that the vector $\mathbf{a} = (0, 0, 1)$ [Koz2], then

$$\begin{cases} I_1\dot{\omega}_1 = \gamma_3\partial_{\gamma_2}U - \gamma_2\partial_{\gamma_3}U \\ I_2\dot{\omega}_2 = \gamma_1\partial_{\gamma_3}U - \gamma_3\partial_{\gamma_1}U \\ (I_1 - I_2)\omega_1\omega_2 + \gamma_2\partial_{\gamma_1}U - \gamma_1\partial_{\gamma_2}U + \mu = 0 \\ \dot{\gamma}_1 = -\gamma_3\omega_2 \\ \dot{\gamma}_2 = \gamma_3\omega_1 \\ \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1 \end{cases} \tag{3.3}$$

The above system has two independent first integrals

$$K_1 = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2) - U(\gamma_1, \gamma_2, \gamma_3)$$
$$K_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$$

By the Jacobi's theorem about the last multiplier, if there exits a third independent first integral K_3 which is functionally independent together with K_1 and K_2, then the Suslov problem is integrable by quadratures [Koz2, Mac]

To determine the integrable cases of the Suslov problem seems interesting the following result which we can prove after straightforward calculations.

Proposition 3.1

Let us suppose that the potential function U in (3.3) is determine as follows

$$U = \frac{1}{2I_1 I_2}(I_1\mu_1^2 + I_2\mu_2^2) - h \tag{3.4}$$

where μ_1, μ_2 are solutions of the partial differential equations

$$\gamma_3(\frac{\partial\mu_1}{\partial\gamma_2} - \frac{\partial\mu_2}{\partial\gamma_1}) - \gamma_2\frac{\partial\mu_1}{\partial\gamma_3} + \gamma_1\frac{\partial\mu_2}{\partial\gamma_3} = 0, \tag{3.5}$$

then the equations (3.3)+(3.4) admits the first integrals

$$I_1\omega_1 = \mu_2, \quad I_2\omega_2 = -\mu_1 \tag{3.6}$$

The aim of this apartat is to propose the Cartesian approach for heavy rigid body in the Suslov case.

Let us suppose that $\mathcal{Q} = SO(3)$, with the Riemann metric

$$G = \begin{pmatrix} I_3 & I_3\cos z & 0 \\ I_3\cos z & (I_1\sin^2 x + I_2\cos^2 x)\sin^2 z + I_3\cos^2 z & (I_1 - I_2)\sin x\cos x\sin z \\ 0 & (I_1 - I_2)\sin x\cos x\sin z & I_1\cos^2 x + I_2\sin^2 x \end{pmatrix}$$

$detG = I_1 I_2 I_3\sin^2 z,$

The given 1-form is the following $\Omega_1 = dx + \cos z dy$. By choosing the 1-form Ω_2, Ω_3 as follow

$$\Omega_2 = dy, \quad \Omega_3 = dz$$

we obtain that $\Upsilon = 1$. Hence the vector field $\mathbf{v}$ is such that

$$\mathbf{v} = \lambda_2(\cos\, z\,\partial_x - \partial_y) - \lambda_3\partial_z = \lambda_2 X_2 - \lambda_3 X_3$$

The equations (3.1) and conditions(3.2) take the form respectively

$$\begin{cases} \dot{x} = \cos z\,\lambda_2, \\ \dot{y} = -\lambda_2, \\ \dot{z} = -\lambda_3, \end{cases} \tag{3.7}$$

$$\Omega_1(rot\mathbf{v}) = \partial_z p_2 - \partial_y p_3 + \cos z\partial_x p_3 = 0 \tag{3.8}$$

where

$$
(3.9) \qquad \begin{cases} rot\mathbf{v} = \dfrac{1}{\sqrt{detG}}(\partial_z p_2 - \partial_y p_3,\, \partial_x p_3,\, -\partial_x p_2) \\ p_k = \dfrac{\partial T}{\partial \dot{x}^k}|_{\dot{\mathbf{x}}=\mathbf{v}} \end{cases}
$$

From (3.9) we deduce that

$$
\begin{cases} -I_1 I_2 \lambda_3 = I_1 p_3 + (I_2 - I_1)(p_3 \cos x + \dfrac{p_2 \sin x}{\sin z}) \\ I_1 I_2 \sin z \lambda_2 = \dfrac{I_1 p_2}{\sin z} + (I_2 - I_1)\sin x (p_3 \cos x + \dfrac{p_2 \sin x}{\sin z}) \end{cases}
$$

By introducing the change

$$
\begin{cases} p_2 = \sin z(\mu_2 \sin x - \mu_1 \cos x) \\ p_3 = \mu_2 \cos x + \mu_1 \sin x, \end{cases}
$$

we obtain that the system (3.7) and equation (3.8) admit the representation respectively

$$
(3.10) \qquad \begin{cases} \dot{x} = \dfrac{\cot z}{I_1 I_2}(I_1 \mu_1 \cos x - I_2 \mu_2 \sin x), \\ \dot{y} = \dfrac{-1}{I_1 I_2 \sin z}(I_1 \mu_1 \cos x - I_2 \mu_2 \sin x) \\ \dot{z} = \dfrac{1}{I_1 I_2}(I_1 \mu_1 \sin x + I_2 \mu_2 \cos x), \end{cases}
$$

$$
(3.11) \qquad \sin x(\sin z \partial_z \mu_2 + \cos z \partial_x \mu_1) + \cos x(\cos z \partial_x \mu_2 - \sin z \partial_z \mu_1 - \partial_y \mu_2) = 0
$$

Clearly,

$$
||\mathbf{v}||^2 = \frac{1}{I_1 I_2}(I_1 \mu_1^2 + I_2 \mu_2^2)
$$

Now we shall study the particular case when (3.11) holds in view of the relations

$$
(3.12) \qquad \begin{cases} \tan z \partial_z \mu_2 = -\partial_x \mu_1 \\ \partial_x \mu_2 = \tan z \partial_z \mu_1 \end{cases}
$$

From the compatibility conditions we obtain the following partial differential equation

$$
(3.13) \qquad \sin^2 z \cos z \frac{\partial^2 \mu_j}{\partial z \partial z} + \cos^3 z \frac{\partial^2 \mu_j}{\partial x \partial x} + \sin\, z \frac{\partial \mu_j}{\partial z} = 0, \quad j = 1, 2
$$

Corollary 3.1

Let μ_1 and μ_2 are solutions of the system (3.12), then the function $F = \mu_1 + \imath\mu_2$ is holomorphic function on the complex variable $w = \gamma_2 + \imath\gamma_1 = e^{\imath x}\sin z$.

In fact, after the change $u = \ln\sin z$ from (3.12) we deduced the Cauchy-Riemann equations

$$\begin{cases} \partial_u\mu_2 = -\partial_x\mu_1 \\ \partial_x\mu_2 = \partial_u\mu_1 \end{cases}$$

i.e., the function

$$F(u + \imath x) = F(\ln w)$$

is an holomorphic function, hence

$$\begin{cases} \mu_1 = \Re F = \partial_{\gamma_2} S = \partial_{\gamma_1}\Psi, \\ \mu_2 = \Im F = \partial_{\gamma_1} S = -\partial_{\gamma_2}\Psi \end{cases}$$

as a consequence we obtain that the functions

$$\begin{cases} I_1\omega_1 = \Re F(\ln(\gamma_2 + \imath\gamma_1)) \\ I_2\omega_2 = -\Im F(\ln(\gamma_2 + \imath\gamma_1)) \end{cases}$$

are first integral of (3.3),(3.4).

Corollary 3.2

If $I_1 = I_2$, then Ψ is a first integral of (3.3) and the function U we determine as follow

$$U = |F(\ln(\gamma_2 + \imath\gamma_1))|^2$$

It is easy to show that the solutions of the equations (3.10) in this case are such that

$$\begin{cases} \displaystyle\int \frac{d(\ln(\gamma_1 + \imath\gamma_2))}{F(\ln(\gamma_1 + \imath\gamma_2))} = \tau - \tau_0 \\ \gamma_3 = \sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)} \\ \displaystyle t = t_0 + \int \frac{d\tau}{\sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)}} \end{cases}$$

Clearly, if μ_1, μ_2 satisfies the Cauchy-Riemann condition then they are solutions of the Laplace equation

$$\frac{\partial^2\mu_j}{\partial u\partial u} + \frac{\partial^2\mu_j}{\partial x\partial x} = 0, \quad j = 1, 2.$$

Hence, if

$$\mu_j = X_j(x)Y_j(u), \quad j = 1, 2 \tag{3.14}$$

then X and Y are solution of the second ordinary differential equation respectively

$$X_j^{''}(x) + \nu^2 X_j(x) = 0, \tag{3.15}$$

and

$$Y_j^{''}(u) - \nu^2 Y_j(u) = 0, \quad j = 1,2 \tag{3.16}$$

where ν is a real constant.

Corollary 3.3

Let μ_1 and μ_2 are solutions of the system (3.13), then

$$\frac{\partial^2 \mu_j}{\partial u \partial u} - \frac{u^2+1}{u-u^3}\frac{\partial \mu_j}{\partial u} + \frac{u^2}{(u^2-1)^2}\frac{\partial^2 \mu_j}{\partial x \partial x} = 0, \quad u = \cos z, \quad j = 1,2$$

If we represent μ_j by the formula (3.14) then X is a solution of the differential equation (3.15). and Y is a solution of the Fuchsian equation

$$Y_j''(u) - \frac{u^2+1}{u-u^3}Y_j'(u) - \frac{\nu^2 u^2}{(u^2-1)^2}Y_j(u) = 0, \quad j = 1,2 \tag{3.17}$$

The proof we obtain after the calculations from (3.13), after the change $u = \cos z$.

Analogously we can prove the following assertion

Corollary 3.4

Let μ_1 and μ_2 are solutions of the system (3.13), then its are solutions of the partial differential equations

$$\frac{\partial^2 \mu_j}{\partial u \partial u} + \frac{1}{u}\frac{\partial \mu_j}{\partial u} + \frac{1}{u^2}\frac{\partial^2 \mu_j}{\partial x \partial x} = 0, \quad u = \sin z, \quad j = 1,2$$

Hence, if (3.14) holds then X satisfies (3.15) and Y is a solution of the Euler ordinary differential equation

$$u^2 Y_j''(u) + uY_j'(u) - \nu^2 Y_j(u) = 0, \quad j = 1,2 \tag{3.18}$$

where ν is real constant.

Proposition 3.2

The functions

$$\begin{cases} I_1\omega_1 = X_2(x)Y_2(u) \\ I_2\omega_2 = -X_1(x)Y_1(u) \end{cases}$$

where X_1, X_2 are solutions of (3.15) and Y_1, Y_2 are solutions of (3.16) or (3.17) or (3.18), are the first integrals of the system (3.3),(3.4).

Denoting by $\gamma_1 = \sin z \sin x, \quad \gamma_2 = \sin z \cos x, \quad \gamma_3 = \cos z$ from (3.10) and (3.11) we obtain the relations respectively

$$\begin{cases} \dot{\gamma_1} = \dfrac{I_1}{I_1 I_2}\mu_1\gamma_3 \\ \dot{\gamma_2} = \dfrac{I_2}{I_1 I_2}\mu_2\gamma_3 \\ \dot{\gamma_3} = \dfrac{-1}{I_1 I_2}(I_1\mu_1\gamma_1 + I_2\mu_2\gamma_2) \end{cases} \tag{3.19}$$

$$\sin z(\gamma_3(\frac{\partial\mu_1}{\partial\gamma_2}-\frac{\partial\mu_2}{\partial\gamma_1})-\gamma_2\frac{\partial\mu_1}{\partial\gamma_3}+\gamma_1\frac{\partial\mu_2}{\partial\gamma_3})-\cos x\partial_y\mu_2-\sin x\partial_y\mu_1=0$$

We shall study only the case when

$$\mu_j=\mu_j(x,z),\ j=1,2$$

Hence, we obtain the equation (3.5).

By compare (3.19) with (3.3) we deduce that

$$I_1\omega_1=\mu_2,\quad I_2\omega_2=-\mu_1 \tag{3.20}$$

Corollary 3.5

Let μ_1, μ_2 are such that The function μ_1, μ_2 :

$$\begin{cases}\mu_1=\dfrac{\partial S(\gamma_1,\gamma_2)}{\partial\gamma_1}+\Psi_1(\gamma_2^2+\gamma_3^2,\gamma_1)\\ \mu_2=\dfrac{\partial S(\gamma_1,\gamma_2)}{\partial\gamma_2}+\Psi_2(\gamma_1^2+\gamma_3^2,\gamma_2)\end{cases}$$

satisfies the equation (3.5).

Corollary 3.7

Let μ_1, μ_2 are such that

$$\mu_j=\frac{\partial S(\gamma_1,\gamma_2)}{\partial\gamma_j},\quad j=1,2$$

then the potential function (3.4) and first integrals (3.6) are respectively

$$\begin{aligned}U&=\frac{1}{2I_1I_2}(I_1(\frac{\partial S}{\partial\gamma_1})^2+I_2(\frac{\partial S}{\partial\gamma_2})^2)-h\\ I_1\omega_1&=\frac{\partial S}{\partial\gamma_2},\\ I_2\omega_2&=-\frac{\partial S}{\partial\gamma_1},\end{aligned}$$

The following particular cases produces the well known integrable cases[Koz2]: The Suslov, Kharlamova-Zabelina and Kozlov subcase.

The Suslov Subcase

If

$$S=C_1\gamma_1+C_2\gamma_2,\quad C_j=const,\ j=1,2$$

then

$$\begin{cases}\mu_1=C_1,\quad \mu_2=C_2\\ U=const.\end{cases}$$

hence we obtain the Suslov subcase

The integration of the equations (3.19) produces the following solutions

$$\begin{cases} \omega_1 = \dfrac{C_2}{I_1}, \quad \omega_2 = -\dfrac{C_1}{I_2} \\ \gamma_1 = \dfrac{C_1 I_1}{\sqrt{I_1^2 C_1^2 + I_2^2 C_2^2}} \sin\beta \sin(\dfrac{\sqrt{I_1^2 C_1^2 + I_2^2 C_2^2}}{I_1 I_2} t + \alpha) + \dfrac{I_2 C_2 \cos\beta}{\sqrt{I_1^2 C_1^2 + I_2^2 C_2^2}} \\ \gamma_2 = \dfrac{C_2 I_2}{\sqrt{I_1^2 C_1^2 + I_2^2 C_2^2}} \sin\beta \sin(\dfrac{\sqrt{I_1^2 C_1^2 + I_2^2 C_2^2}}{I_1 I_2} t + \alpha) - \dfrac{I_1 C_1 \cos\beta}{\sqrt{I_1^2 C_1^2 + I_2^2 C_2^2}} \\ \gamma_3 = \sin\beta \cos((\sqrt{\dfrac{I_1^2 C_1^2 + I_2^2 C_2^2}{I_1 I_2}} t + \alpha) \end{cases}$$

where C_1, C_2, α, β, are the arbitrary real constants.

The Kharlamova-Zabelina Subcase

If

$$S = \frac{2}{3\sqrt{I_1 C_1^2 + I_2 C_2^2}} (\sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2})^3 + \frac{C C_2 I_2}{C_1^2 I_1 + C_2^2 I_2}\gamma_1 - \frac{C C_1 I_1}{C_1^2 I_1 + C_2^2 I_2}\gamma_2$$

where $\tilde{h}$, C_1, C_2, C are arbitrary constants, then

$$\begin{cases} \mu_1 = \dfrac{C_1}{\sqrt{I_1 C_1^2 + I_2 C_2^2}} \sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} + \dfrac{C C_2 I_2}{C_1^2 I_1 + C_2^2 I_2} \\ \mu_2 = \dfrac{C_2}{\sqrt{I_1 C_1^2 + I_2 C_2^2}} \sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} - \dfrac{C C_1 I_1}{C_1^2 I_1 + C_2^2 I_2} \\ U = \tilde{h} + C_1\gamma_1 + C_2\gamma_2 \end{cases}$$

As a consequence we deduce the Kharlamova-Zabelina subcase.

The solutions of the equation (3.19) give the following solutions

$$\begin{cases} I_1\omega_1 = \dfrac{C_1}{\sqrt{I_1 C_1^2 + I_2 C_2^2}} \sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} + \dfrac{C C_2 I_2}{C_1^2 I_1 + C_2^2 I_2}, \\ I_2\omega_2 = -(\dfrac{C_2}{\sqrt{I_1 C_1^2 + I_2 C_2^2}} \sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} - \dfrac{C C_1 I_1}{C_1^2 I_1 + C_2^2 I_2}), \\ U = \tilde{h} + C_1\gamma_1 + C_2\gamma_2 \\ \gamma_j = a_1(\tau - C_3)^2 + b_j(\tau - C_4) + d_j = \gamma_j(\tau, C_1, C_2, C_3, C_4), \quad j = 1, 2 \\ \gamma_3 = \sqrt{1 - \gamma_1^2(\tau, C_1, C_2, C_3, C_4) - \gamma_2^2(\tau, C_1, C_2, C_3, C_4)} \equiv \sqrt{P_4(\tau, C_1, C_2, C_3, C_4)} \\ t = t_0 + \dfrac{I_1 I_2}{2} \displaystyle\int \dfrac{d\tau}{\sqrt{P_4(\tau, C_1, C_2, C_3, C_4)}} \end{cases}$$

where

$$a_j = \frac{I_j C_j}{4}, \quad b_j = \frac{C I_1 I_2 C_1 C_2}{C_j (I_1 C_1^2 + I_2 C_2^2)}, \quad d_j = -\frac{\tilde{h} I_j C_j}{I_1 C_1^2 + I_2 C_2^2},$$

P_4 is a polynomial of four degree in τ.

Kozlov Subcase

If we suppose that $I_1 = I_2$ and

$$\begin{cases} S = -2C \arctan \dfrac{\gamma_1}{\gamma_2} + \displaystyle\int D(\gamma_1^2 + \gamma_2^2) d(\gamma_1^2 + \gamma_2^2) \\ (D(u))^2 = \dfrac{hu^2 + \sqrt{1-u}u - C^2}{u^2} \end{cases}$$

where h and C are arbitrary real constant.

Hence,

$$\begin{cases} \mu_1 = -\dfrac{\gamma_2 C}{\gamma_1^2 + \gamma_2^2} + \gamma_1 D(\gamma_1^2 + \gamma_2^2) \\ \mu_2 = \dfrac{\gamma_1 C}{\gamma_1^2 + \gamma_2^2} + \gamma_2 D(\gamma_1^2 + \gamma_2^2) \\ U = -h + \sqrt{1 - \gamma_1^2 - \gamma_2^2} = -h + \gamma_3 \end{cases}$$

which correspond to the Kozlov subcase.

The equations (3.10) in this case take the form:

$$\begin{cases} \dot{x} = \dfrac{C \cos z}{\sin^2 z} \\ \dot{y} = \dfrac{-C}{\sin^2 z} \\ \dot{z} = \dfrac{(\gamma_1^2 + \gamma_2^2) D(\gamma_1^2 + \gamma_2^2)}{\sin z} \end{cases} \tag{3.21}$$

which are easy to integrate.

The solutions of the equation of motions are:

$$\begin{cases} \omega_1 = \dfrac{\gamma_1 C}{\gamma_1^2+\gamma_2^2} + \gamma_2 D(\gamma_1^2+\gamma_2^2) \\ \omega_2 = \dfrac{\gamma_2 C}{\gamma_1^2+\gamma_2^2} - \gamma_1 D(\gamma_1^2+\gamma_2^2) \\ x = x_0 + C\displaystyle\int \frac{\gamma_3 d\gamma_3}{(1-\gamma_3^2)^2 D(1-\gamma_3^2)} = x_0 + C\int \frac{\gamma_3 d\gamma_3}{\sqrt{(1-\gamma_3^2)P_4(\gamma_3,h,C)}} \\ y = y_0 - C\displaystyle\int \frac{d\gamma_3}{(1-\gamma_3^2)^2 D(1-\gamma_3^2)} = y_0 - C\int \frac{d\gamma_3}{\sqrt{(1-\gamma_3^2)P_4(\gamma_3,h,C)}} \\ t = t_0 + I_1 I_2 \displaystyle\int \frac{d\gamma_3}{\sqrt{P_4(\gamma_3,h,C)}} \\ P_4(\gamma_3,h,C) \equiv h\gamma_3^4 - 2\gamma_3^3 - 2h\gamma_3^2 + 2\gamma_3 + h - C^2 \end{cases}$$

where x_0, y_0, h, C, t_0 are arbitrary constants

Corollary 3.6

Let μ_1, μ_2 are the functions:

$$\begin{cases} \mu_1 = \Psi_1(\gamma_2^2+\gamma_3^2,\gamma_1) \\ \mu_2 = \Psi_2(\gamma_1^2+\gamma_3^2,\gamma_2) \end{cases}$$

then the solutions of (3.19) are the following functions:

$$\begin{cases} \displaystyle\int \frac{d\gamma_j}{F_j(\gamma_j)} = \frac{I_j}{I_1 I_2}(\tau-\tau_0), \quad j=1,2 \\ \gamma_3 = \sqrt{1-\gamma_1^2(\tau)-\gamma_2^2(\tau)} \\ t = t_0 + \displaystyle\int \frac{d\tau}{\sqrt{1-\gamma_1^2(\tau)-\gamma_2^2(\tau)}} \end{cases}$$

where

$$F_1(\gamma_1) = \Psi_1(\gamma_2^2+\gamma_3^2,\gamma_1)|_{\gamma_2^2+\gamma_3^2=1-\gamma_1^2}$$
$$F_2(\gamma_2) = \Psi_2(\gamma_1^2+\gamma_3^2,\gamma_2)|_{\gamma_1^2+\gamma_3^2=1-\gamma_2^2}$$

As a particular case we obtain the Tisserand Subcase.

Tisserand Subcase

The interesting solution of the equation (3.20) are

$$\begin{cases} \mu_1 = \sqrt{h_1 + a_1(\gamma_3^2+\gamma_2^2) + b_1\gamma_1^2 + f_1(\gamma_1)} \equiv \Psi_1(\gamma_2^2+\gamma_3^2,\gamma_1) \\ \mu_2 = \sqrt{h_2 + a_2(\gamma_3^2+\gamma_1^2) + b_2\gamma_2^2 + f_2(\gamma_2)} \equiv \Psi_2(\gamma_1^2+\gamma_3^2,\gamma_2) \end{cases}$$

which produce the following potential function U :

$$U = I_1h_1 + I_2h_2 + (I_1b_1 + I_2a_2)\gamma_1^2 + (I_1a_1 + I_2b_2)\gamma_2^2 + (I_1a_1 + I_2a_2)\gamma_3^2 + I_1f_1(\gamma_1) + I_2f_2(\gamma_2)$$

where a_j, b_j, h_j, $j = 1, 2$ are arbitrary real constants and f_j, $j = 1, 2$ are arbitrary functions.

The case when $f_j(\gamma_j) = \alpha_j\gamma_j$, $j = 1, 2$ was studied in [Okuneba], where α_j, $j = 1, 2$ are real constants.

The case when $f_j = 0$, $j = 1, 2$ is well known as Tisserands case [Koz2].

After integration the equation (3.19) in the Tisserand case we obtain the following solutions

$$\begin{cases} I_1\omega_1 = \sqrt{h_2 + a_2(\gamma_3^2 + \gamma_1^2) + b_2\gamma_2^2} \\ I_2\omega_2 = -\sqrt{h_1 + a_1(\gamma_3^2 + \gamma_2^2) + b_1\gamma_1^2} \\ \gamma_1 = \sqrt{\dfrac{h_1 + a_1}{a_1 - b_1}}\sin(\sqrt{a_1 - b_1}I_1\tau + C_1) = \gamma_1(\tau) \\ \gamma_2 = \sqrt{\dfrac{h_2 + a_2}{a_2 - b_2}}\sin(\sqrt{a_2 - b_2}I_2\tau + C_2) = \gamma_2(\tau) \\ \gamma_3 = \sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)} \\ t = t_0 + I_1I_2\displaystyle\int \frac{d\tau}{\sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)}} \end{cases}$$

HEAVY RIGID BODY IN THE VESELOV CASE

In this example we study the problem of non-holonomic dynamics formulated by Veselov in [Veselov] which in certain sense is opposite to the Suslov problem. In this problem we consider the rotational motion of a rigid body around a fixed point and subject to the non-holonomic constraints

$$(\gamma, \omega) \equiv \dot{y} + \cos z\dot{x} = 0 \qquad (3.22)$$

Suppose the body rotates in an force field with potential $U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers we write the equations of motion in the form

$$\begin{cases} I\dot{\omega} = [I\omega \times \omega] + [\gamma \times \dfrac{\partial U}{\partial \gamma}] + \lambda\gamma \\ \dot{\gamma} = [\gamma \times \omega] \end{cases}$$

where I is a matrix such that $I = diag(I_1, I_2, I_3)$.

The Cartesian approach for this system produces the following equations:

$$
\begin{cases} \dot{x} = \lambda_2 \\ \dot{y} = -\cos z \lambda_2 \\ \dot{z} = \lambda_3 \end{cases} \tag{3.23}
$$

and

$$
\frac{\partial p_3}{\partial x} - \frac{\partial p_1}{\partial z} + \cos z (\frac{\partial p_2}{\partial z} - \frac{\partial p_3}{\partial y}) = 0 \tag{3.24}
$$

where

$$
\begin{cases} p_1 = I_3 \sin^2 z \lambda_2 \\ p_2 = (I_3 - I_1 + (I_1 - I_2)\cos^2 x) \cos z \sin^2 z \lambda_2 + (I_1 - I_2) \cos x \sin x \sin z \lambda_3 \\ p_3 = (I_2 \sin^2 x + I_1 \cos^2 x)\lambda_3 + (I_2 - I_1) \sin x \cos x \sin z \cos z \lambda_2 \end{cases} \tag{3.25}
$$

Hence

$$
\begin{cases} \omega_1 = \gamma_2 \dfrac{\lambda_3}{\sin z} - \gamma_1 \gamma_3 \lambda_2 \\ \omega_2 = -\gamma_1 \dfrac{\lambda_3}{\sin z} - \gamma_2 \gamma_3 \lambda_2 \\ \omega_3 = \sin^2 z \lambda_2 \end{cases} \tag{3.26}
$$

Clearly, in this case

$$
\begin{aligned} ||\mathbf{v}||^2 = & (I_3 \sin^2 z + (I_1 \sin^2 x + I_2 \cos^2 x)\cos^2 z) \sin^2 z \lambda_2^2 + \\ & (I_2 \sin^2 x + I_1 \cos^2 x)\lambda_3^2 + 2(I_2 - I_1) \cos x \sin x \cos z \sin z \lambda_2 \lambda_3 \end{aligned} \tag{3.27}
$$

It is interesting to observe that from (3.25) after the change

$$
\begin{cases} \lambda_2 \sin z = \dfrac{\cos x \, \mu_2}{I_3 + ((I_2 - I_3)\cos^2 z} - \dfrac{\sin x \, \mu_1}{I_3 + ((I_1 - I_3)\cos^2 z} \\ \lambda_3 = \dfrac{\cos z \cos x \, \mu_1}{I_3 + ((I_1 - I_3)\cos^2 z} + \dfrac{\cos z \sin \, x \, \mu_2}{I_3 + ((I_2 - I_3)\cos^2 z} \end{cases} \tag{3.28}
$$

we obtain that the equations (3.24) can be rewritten as follows:

$$
\begin{cases} \dot{x} = \dfrac{\cos x \, \mu_2}{\sin z(I_3 + ((I_2 - I_3)\cos^2 z)} - \dfrac{\sin x \, \mu_1}{\sin z(I_3 + ((I_1 - I_3)\cos^2 z)} \\ \dot{y} = -\dfrac{\cos z \cos x \, \mu_2}{\sin z(I_3 + ((I_2 - I_3)\cos^2 z)} + \dfrac{\cos z \sin x \, \mu_1}{\sin z(I_3 + (I_1 - I_3)\cos^2 z)} \\ \dot{z} = \dfrac{\cos z \cos x \, \mu_1}{I_3 + ((I_1 - I_3)\cos^2 z} + \dfrac{\cos z \sin \, x \, \mu_2}{I_3 + ((I_2 - I_3)\cos^2 z} \end{cases} \tag{3.29}
$$

and by require that

$$\mu_j = \mu_j(x,z), \quad j = 1,2$$

we deduce the following equation

$$\sin x\Big(\frac{\partial_x \mu_2}{1+(\alpha-1)\sin^2 z} + \tan z \partial_z \mu_1\Big) + \cos x\Big(\frac{\partial_x \mu_1}{1+(\beta-1)\sin^2 z} - \tan z \partial_z \mu_2\Big) = 0, \tag{3.30}$$

where $\alpha = \frac{I_3}{I_2}$, $\beta = \frac{I_3}{I_1}$. If $\alpha = \beta = 1$ then this equation coincide with (3.11).

Proposition 3.2

Let μ_1, μ_2 are solutions of (3.30). Then the equations of motions of the heavy rigid body in Veselov case admit the following first integrals:

$$\begin{cases} \omega_1 = \dfrac{\gamma_3 \mu_1}{I_3 + ((I_1 - I_3)\cos^2 z} \\ \omega_2 = \dfrac{-\gamma_3 \mu_2}{(I_3 + ((I_2 - I_3)\cos^2 z)} \\ \omega_3 = \dfrac{\gamma_2 \mu_2}{(I_3 + ((I_2 - I_3)\cos^2 z)} - \dfrac{\gamma_1 \mu_1}{(I_3 + ((I_1 - I_3)\cos^2 z)} \end{cases} \tag{3.31}$$

Hence, the Cartesian approach for the Veselov case produce the following first integrals

1.

$$\begin{cases} \omega_1 = \dfrac{\gamma_3 C_1}{I_3 + ((I_1 - I_3)\cos^2 z} \\ \omega_2 = \dfrac{-\gamma_3 C_2}{(I_3 + ((I_2 - I_3)\cos^2 z)} \\ \omega_3 = \dfrac{\gamma_2 C_2}{(I_3 + ((I_2 - I_3)\cos^2 z)} - \dfrac{\gamma_1 C_1}{(I_3 + ((I_1 - I_3)\cos^2 z)} \end{cases}$$

For the symmetrical case, i.e., $I_1 = I_2$ we obtain the following trajectories of the body

$$\begin{cases} y = y_0 - \arcsin\Big(\dfrac{\sin(x+\alpha)}{1+C_3^2}\Big) \\ z = z_0 + \arctan \dfrac{C_3}{\cos(x+\alpha)} \\ t = t_0 + C_3^3 \displaystyle\int \dfrac{(I_3\cos^2(x+\alpha) + I_2 C_3^2)dx}{(\sqrt{C_3^2\cos^2(x+\alpha) + \cos^4(x+\alpha)})^3} \end{cases}$$

where C_1, C_2, C_3, are arbitrary constants.

2.

$$\begin{cases} I_1\omega_1 = \dfrac{\gamma_3\gamma_2 C_2}{\sqrt{((\beta-1)\sin^2 z + 1)^3}} \\ I_2\omega_2 = \dfrac{-\gamma_3\gamma_1 C_1}{\sqrt{((\alpha-1)\sin^2 z + 1))^3}} \\ I_3\omega_3 = \dfrac{\alpha\gamma_2\gamma_1 C_1}{\sqrt{((\alpha-1)\sin^2 z + 1)^3}} - \dfrac{\beta\gamma_1\gamma_2 C_2}{\sqrt{((\beta-1)\sin^2 z + 1)^2}} \end{cases}$$

For the symmetrical case, i.e., $I_1 = I_2$ we obtain the following trajectories

$$\begin{cases} y = y_0 - \int \frac{dx}{\sqrt{1+\Psi^2(x, C_1, C_2, C_3)}} \\ z = z_0 + \arctan \Psi(x, C_1, C_2, C_3) \\ t = t_0 + \int \frac{(I_3 + I_2\Psi^2(x, C_1, C_2, C_3))\Psi^3(x, C_1, C_2, C_3)dx}{\cos(x+\alpha)\sqrt{(1+\Psi^2(x, C_1, C_2, C_3))^3}} \\ \Psi(x, C_1, C_2, C_3) = C_3(\frac{\cos^\alpha x}{\sin^\beta x}), \quad \alpha = \frac{C_1}{C_1 - C_2}, \beta = \frac{C_2}{C_1 - C_2} \end{cases}$$

where C_1, C_2, C_3, y_0, z_0, t_0 are arbitrary constants.

Corollary 3.8

If $I_1 = I_2$, then the functions

$$\begin{cases} \mu_2 = (1 + (\beta - 1)\sin^2 z)(\frac{C\cos x}{\sin z\sqrt{1 + (\beta-1)\sin^2 z}} + \frac{\lambda_3(z)\sin x}{\cos z}) \\ \mu_1 = (1 + (\beta - 1)\sin^2 z)(-\frac{C\sin x}{\sin z\sqrt{1 + (\beta-1)\sin^2 z}} + \frac{\lambda_3(z)\cos x}{\cos z}) \end{cases}$$

are solutions of the equation (3.28) where C is an arbitrary constant and λ_3 is an arbitrary function on z.

Hence we easily deduced the proof of the following result

Corollary 3.9

The system (3.24)for the symmetrical case $I_1 = I_2$ admits the following solutions

$$\begin{cases} x = x_0 + C\int \frac{dz}{(1-\cos^2 z)\lambda_3(z)\sqrt{I_3 + (I_1 - I_3)\cos^2 z}} \\ y = y_0 - C\int \frac{\cos z dz}{(1-\cos^2 z)\lambda_3(z)\sqrt{I_3 + (I_1 - I_3)\cos^2 z}} \\ t = t_0 - \int \frac{dz}{\lambda_3(z)} \end{cases}$$

In this case there exist the first integral

$$(I_3\sin^2 z + I_1\cos^2 z)\omega_3^2 = C^2.$$

Clearly, the equation (3.30) holds if

$$\begin{cases} \frac{\partial_x\mu_2}{1 + (\alpha-1)\tau^2} + \tau\partial_\tau\mu_1 = 0 \\ \frac{\partial_x\mu_1}{1 + (\beta-1)\tau^2} - \tau\partial_\tau\mu_2 = 0 \end{cases}$$

Hence the functions ξ and η satisfies the equations

$$(\partial^2_{xx}+(1+3(\alpha-1)\tau^2)(\tau+(\beta-1)\tau^3)\partial_\tau+(\tau+(\alpha-1)\tau^3)(\tau+(\beta-1)\tau^3)\partial^2_{\tau\tau})\mu_1=0$$
$$(\partial^2_{xx}+(1+3(\beta-1)\tau^2)(\tau+(\alpha-1)\tau^3)\partial_\tau+(\tau+(\alpha-1)\tau^3)(\tau+(\beta-1)\tau^3)\partial^2_{\tau\tau}))\mu_2=0$$

By represent μ_1 and μ_2 as follows

$$\mu_1=X_1(x)Y_1(\tau),\quad \mu_2=X_2(x)Y_2(\tau)$$

we obtain that X_j, Y_j, $j=1,2$ are such that

$$X_1{}''(x)+a^2X_1(x)=0,\quad a=const.$$
$$Y_1^{''}(\tau)+\frac{(1+3(\alpha-1)\tau^2)}{\tau+(\alpha-1)\tau^3}Y_1^{'}(\tau)-\frac{a^2}{(\tau+((\alpha-1)\tau^3)(\tau+(\beta-1)\tau^3))}Y_1(\tau)=0$$

and

$$X_2{}''(x)+a^2X_2(x)=0\quad a=const.$$
$$Y_1^{''}(\tau)+\frac{(1+3(\beta-1)\tau^2)}{\tau+(\beta-1)\tau^3}Y_1^{'}(\tau)-\frac{a^2}{(\tau+(\alpha-1)\tau^3)(\tau+(\beta-1)\tau^3)}Y_1(\tau)=0$$

Finally it is interesting to observe that the construction the Cartesian approach for the Federov case [Federov], i.e.,

$$(\omega,\gamma)=a$$

it is necessary in the above example make the change $y\to y+at$, $a=const.$. Hence we obtain that

(3.32)
$$\begin{cases}\omega_1=\gamma_2\dfrac{\lambda_3}{\sin z}-\gamma_1\gamma_3\lambda_2-a\gamma_1\\ \omega_2=-\gamma_1\dfrac{\lambda_3}{\sin z}-\gamma_2\gamma_3\lambda_2-a\gamma_2\\ \omega_3=\sin^2 z\lambda_2-a\gamma_3\end{cases}$$

The equations generated by the Cartesian approach in this case are the following

(3.33)
$$\begin{cases}\dot{x}=\lambda_2\\ \dot{y}=-\cos z\lambda_2+a\\ \dot{z}=\lambda_3\end{cases}$$

and

(3.34)
$$\frac{\partial p_3}{\partial x}-\frac{\partial p_1}{\partial z}+\cos z(\frac{\partial p_2}{\partial z}-\frac{\partial p_3}{\partial y}))=0$$

where

$$\begin{cases} p_1 = I_3 \sin^2 z \lambda_2 \\ p_2 = (I_3 - I_1 + (I_1 - I_2)\cos^2 x)\cos z \sin^2 z \lambda_2 + (I_1 - I_2)\cos x \sin x \sin z \lambda_3 + \\ \quad a((I_1 \sin^2 x + I_2 \cos^2 x)\sin^2 z + I_3 \cos^2 z) \\ p_3 = (I_2 \sin^2 x + I_1 \cos^2 x)\lambda_3 + (I_2 - I_1)\sin x \cos x \sin z \cos z \lambda_2 \end{cases}$$

The equation (3.25) can be represented as follow

$$(3.35) \qquad \frac{\partial P_3}{\partial x} - \frac{\partial P_1}{\partial z} + \cos z (\frac{\partial P_2}{\partial z} - \frac{\partial P_3}{\partial y}) + 2a \cos^2 z \sin z (I_1 \sin^2 x + I_2 \cos^2 x - I_3) = 0$$

where

$$\begin{cases} P_1 = p_1 \\ P_2 = p_2 - a((I_1 \sin^2 x + I_2 \cos^2 x)\sin^2 z + I_3 \cos^2 z) \\ P_3 = p_3 \end{cases}$$

In particular if $P_j = P_j(x, z)$ and $I_1 = I_2$ we obtain that the above equations takes the form

$$(3.36) \qquad \begin{cases} \frac{I_3}{I_2} \frac{\partial \lambda_3}{\partial x} - \nu \frac{\partial}{\partial z}(\nu \sin^2 z \lambda_2) + 2a \cos^2 z \sin z (I_2 - I_3) = 0, \\ \nu = \sqrt{I_3 \sin^2 z + I_1 \cos^2 z} \end{cases}$$

The functions λ_2, λ_3 :

$$\begin{cases} \lambda_3 = \lambda_3(z) \\ \sqrt{I_3 \sin^2 z + I_2 \cos^2 z} \sin^2 z \lambda_2 = 2a(I_2 - I_3) \int \frac{\cos z^2 \sin z dz}{\sqrt{I_3 \sin^2 z + I_2 \cos^2 z}} + C_1 \end{cases}$$

where C_1, is an arbitrary constant, are solutions of (3.36). Hence, we obtain the existence the first integral

$$(I_3 \sin^2 z + I_2 \cos^2 z)(\omega_3 + a \cos z)^2 = (C_1 + a \int \sin z \sqrt{I_3 \sin^2 z + I_2 \cos^2 z} dz)^2$$

THE CHAPLIGUIN-CARATHEODORY SLEIGH

We shall now analyse one of the most classical nonholonomic systems : Chapliguin-Carathodory's sleigh [NF]. The idealized sleigh is a body that has three points of contact with the plane. Two of them slide freely but the third, A, behaves like a knife edge subjected to a constraining force **R** which does not allow transversal velocity. More precisely, let yoz be an inertial frame and $\xi A \eta$ a frame moving with the sleigh. Take as generalized coordinates the Cartesian coordinates of the center of mass C of the sleigh and the angle x

between the y and the ξ axis. The reaction force $\mathbf{R}$ against the runners is exerted laterally at the point of application A in such a way that the η component of the velocity is zero. Hence, one has the constrained system $\mathcal{M}$ with the configuration space $X = S^1 \times \mathrm{R}^2$, with the kinetic energy $T = \frac{m}{2}(\dot{y}^2 + \dot{z}^2) + \frac{I_c}{2}\dot{x}^2$, and with the constraint $\epsilon\dot{x} + \sin x\dot{y} - \cos x\dot{z} = 0$, where m is the mass of the system and I_c is the moment of inertia about a vertical axis through C and $\epsilon = |AC|$. Observe that the "javelin" (or arrow or Chapliguin's skate) is a particular case of this mechanical system and can be obtained when $\epsilon = 0$

To apply the Cartesian approach for this system, first we introduce the 1-form Ω_1, Ω_2, Ω_3 in such a way that the determinant $\Upsilon \neq 0$. In this subcase, we achieve this condition if

$$\Omega_1 = \epsilon dx + \sin x dy - \cos x dz, \quad \Omega_2 = \sin x dz + \cos x dy, \quad \Omega_3 = dx.$$

Under these restrictions we obtain that $\Upsilon = 1$ and it is easy to show that the vector field $\mathbf{v}$ takes the form:

$$\mathbf{v} = \lambda_3(\partial_x + \epsilon \sin x\partial_y + \epsilon \cos x\partial_z) - \lambda_2(\cos x\partial_y - \sin x\partial_z) = \lambda_3 X_3 + \lambda_2 X_2$$

It is easy to show that the vector field X_1, X_2, X_3 :

$$\begin{cases} X_3 = \partial_x + \epsilon \sin x\partial_y + \epsilon \cos x\partial_z \\ X_2 = \cos x\partial_y - \sin x\partial_z \\ X_1 = \partial_x \\ [X_2, X_3] = -[X_1, X_3] = \epsilon X_2, \quad [X_1, X_2] = -\dfrac{1}{\epsilon}(X_3 - X_1) \end{cases}$$

if $\epsilon \neq 0$ and

$$\begin{cases} X_3 = \partial_x \\ X_2 = \cos x\partial_y - \sin x\partial_z \\ X_1 = \cos x\partial_z + \sin x\partial_y \\ [X_2, X_3] = [X_1, X_2] = -X_1, \quad [X_3, X_2] = 0 \end{cases}$$

if $\epsilon = 0$, generated a three dimensional Lie-algebra

If we introduce the vector fields respectively:

$$Z = \epsilon X_2, \quad X = X_1, \quad Y = X_1 - X_3$$

and

$$X = X_3, \quad Y = X_1, \quad Z = X_2$$

we deduce the Bianchi representation with $a = b_1 = 0$, $b_3 = -b_2 = 1$ for the both case .

The Cartessian approach in this case produce the differential equations

$$\begin{cases} \dot{x} = \lambda_3(x, y, z, \epsilon) \\ \dot{y} = \lambda_2(x, y, z, \epsilon)\cos x - \epsilon\lambda_3 \sin x \\ \dot{z} = \lambda_2(x, y, z, \epsilon)\sin x + \epsilon\lambda_3 \cos x \end{cases} \tag{3.37}$$

where λ_2, λ_3 are solutions of the partial differential equations

$$\sin x((J_C+\epsilon^2 m)\partial_z\lambda_3+\epsilon m\partial_y\lambda_2)+\cos x((J_C+\epsilon^2 m)\partial_y\lambda_3- \epsilon m\partial_z\lambda_2)-m(\partial_x\lambda_2-\epsilon\lambda_3)=0 \tag{3.38}$$

Hence, for the arrow ($\epsilon=0$) we have

$$\begin{cases} \dot{x}=\lambda_3(x,y,z,0) \\ \dot{y}=\lambda_2(x,y,z,0)\cos x \\ \dot{z}=\lambda_2(x,y,z,0)\sin x \end{cases} \tag{3.39}$$

$$J_C(\sin x\partial_z\lambda_3+\cos x\partial_y\lambda_3)-m\partial_x\lambda_2=0 \tag{3.40}$$

Clearly, the equation (3.38) holds if

$$\begin{cases} \partial_y\lambda_3=\dfrac{\epsilon m}{J_C+\epsilon^2 m}\partial_z\lambda_2 \\ \partial_z\lambda_3=-\dfrac{\epsilon m}{J_C+\epsilon^2 m}\partial_y\lambda_2 \\ \partial_x\lambda_2=a\lambda_3 \end{cases} \tag{3.41}$$

After some calculations we can prove that the functions

$$\begin{cases} \lambda_2=\cos\alpha\, V_1(y,z,\epsilon)+\sin\alpha\, V_2(y,z,\epsilon)+a\displaystyle\int K(x,\epsilon)dx \\ \lambda_3=\dfrac{am}{J_C+a^2 m}\Big(cos\alpha\, V_2(y,z,a)-\sin\alpha\, V_1(y,z,\epsilon)\Big)+K(x,\epsilon), \\ \alpha=\dfrac{\epsilon^2 mx}{J_C+\epsilon^2 m} \end{cases} \tag{3.42}$$

are solutions of (3.38), where K is an arbitrary function and V_1, V_2 are functions which satisfy the Cauchy-Riemann conditions:

$$\begin{cases} \partial_y V_1(y,z,\epsilon)=\partial_z V_2(y,z,\epsilon) \\ \partial_z V_1(y,z,\epsilon)=-\partial_y V_2(y,z,\epsilon). \end{cases}$$

The aim of the assertions below is to study the differential equations generated by the vector field $\mathbf{v}$:

From the above it is easy to show the following consequences [Sad].

Corollary 3.10

The all trajectories of the Chapliguin skate ($\epsilon=0$) and the Chapliguin- Caratheodory sleigh by inertia can be obtained from (3.39), (3.40).

In fact, for the case when $\epsilon = 0$ the classical approach gives the following equations of motion

$$\begin{cases} \ddot{x} = 0 \\ \ddot{y} = mg + \sin x\mu \\ \ddot{z} = -\cos x\mu \\ \sin x\dot{y} - \cos x\dot{z} = 0 \end{cases}$$

Hence,by derivation we obtain

$$\frac{d}{dt}(\frac{\dot{z}}{\sin x}) = g\cos x$$

as a consequence,

$$\begin{cases} \dot{x} = C_0, \quad C_0 \neq 0 \\ \dot{y} = (\frac{g\sin x}{C_0} + C_1)\cos x \\ \dot{z} = (\frac{g\sin x}{C_0} + C_1)\sin x \end{cases}$$

or,

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = (gt\cos x_0 + C_1)\cos x_0 \\ \dot{z} = (gt\cos x_0 + C_1)\sin x_0 \end{cases}$$

Clearly, the solutions of these equations coincide with the solutions of (3.39), (3.40) with the subsidiary conditions

$$J_C\lambda_3^2 + m\lambda_2^2 = 2(U + h).$$

are particular case of the equations obtained from the Cartesian approach.

Let us suppose that

$$\lambda_j = \lambda_j(x, \epsilon), \quad j = 1, 2$$

then the all trajectories of the Chapliguin- Caratheodory sleigh under the action of the potential field of force with potential U :

$$2(U + h) = m\lambda_2^2(x, \epsilon) + (m\epsilon^2 + J_C)\lambda_3^2(x, \epsilon)$$

can be obtained from the formula

(3.43)
$$\begin{cases} y = y_0 + \int \frac{(\lambda_2(x, \epsilon)\cos x - \epsilon\lambda_3\sin x)dx}{\lambda_3(x, \epsilon)} \\ z = z_0 - \int \frac{(\lambda_2(x, y, z, \epsilon)\sin x - \epsilon\lambda_3\cos x)dx}{\lambda_3(x, \epsilon)} \\ t = t_0 + \int \frac{dx}{\lambda_3(x, \epsilon)} \end{cases}$$

For the Chapliguin- Caratheodory sleigh by inertia from the classical approach we deduce the following equations

$$\begin{cases} J_C\ddot{x} = \epsilon\mu \\ m\ddot{y} = \sin x\mu \\ m\ddot{z} = -\cos x\mu \\ \epsilon\dot{x} + \sin x\dot{y} - \cos x\dot{z} = 0 \end{cases}$$

Hence, after straightforward calculations we obtain the system

$$\begin{cases} \dot{x} = qC_0\cos(q\epsilon x + C), \quad q^2 = \dfrac{m}{J_C + m\epsilon^2} \\ \dot{y} = C_0(\sin(q\epsilon x + C)\cos x - q\epsilon\cos(q\epsilon x + C)\sin x) \\ \dot{z} = C_0(\sin(q\epsilon x + C)\sin x + q\epsilon\cos(q\epsilon + C)\cos x) \end{cases}$$

which are particular case of the equations (3.37)with

$$\lambda_2 = C_0\cos(q\epsilon x + C), \quad \lambda_3 = C_0 q\cos(q\epsilon x + C)$$

Evident that in this case

$$2||\mathbf{v}||^2 = (J_C + m\epsilon^2)\lambda_3^2(x, \epsilon\,) + m\lambda_2^2(x, \epsilon\,) \equiv mC_0^2$$

4. CARTESIAN APPROACH FOR NONHOLONOMIC SYSTEM WITH FOUR DEGREE OF FREEDOM AND TWO CONSTRAINTS .

In this apartat we illustrate the Cartesian approach for mechanical system with four degree of freedom and two constraints in the well known example which we call the Gantmacher system [Gantmacher]

GANTMACHER SYSTEM

Let

$$\mathcal{Q} = \mathrm{R}^4, \quad L = \frac{1}{2}\sum_{j=1}^{4}\dot{u}_j^2 - gu_3$$

be the configuration space and kinetic energy of the mechanical system which is under the constraints

$$\begin{cases} u_1\dot{u}_1 + u_2\dot{u}_2 = 0 \\ u_1\dot{u}_3 - u_2\dot{u}_4 = 0 \end{cases}$$

To construct the Cartesian approach in this case we firstly determine the 1-forms Ω_3, Ω_4 as follow

$$\Omega_3 = u_1du_2 - u_2du_1, \quad \Omega_4 = u_2du_3 + u_1du_4$$

hence we obtain

$$
\begin{aligned}
&\Upsilon = (u_1^2 + u_2^2)^2 \\
&\mathbf{v} = \nu_3(u_1\partial_2 - u_2\partial_1) + \nu_4(u_2\partial_3 + u_1\partial_4) \equiv \nu_3 X_3 + \nu_4 X_2 \\
&\nu_j = \lambda_j(u_1^2 + u_2^2), \quad j = 1, 2
\end{aligned}
$$

The conditions (2.3)

$$\Lambda_3 = \Lambda_4 = 0$$

in this case hold iff $u_1^2 + u_2^2 \neq 0$ and

$$u_2\partial_1\nu_4 - u_1\partial_2\nu_4 + u_2\partial_3\nu_3 + u_1\partial_4\nu_3 = 0 \tag{4.1}$$

The vector fields X_3, X_4 are contain in three dimensional Lie algebra:

$$
\begin{cases}
[X_3, X_2] = u_1\partial_3 - u_2\partial_4 \equiv X_1 \\
[X_1, X_2] = 0, \quad [X_1, X_3] = -X_2
\end{cases}
$$

which correspond to the Bianchi representation with $a = b_1 = 0$, $b_3 = -b_2 = 1$.

It is easy to show that the functions ν_3, ν_4 :

$$\nu_3 = g_3(u_1^2 + u_2^2), \qquad \nu_4 = \sqrt{\frac{2(-gu_3 + h)}{(u_1^2 + u_2^2)} - g_3^2(u_1^2 + u_2^2)},$$

where g, h are constants, are solutions of (4.1) as a consequence

$$2||\mathbf{v}(x)||^2 = (u_1^2 + u_2^2)(\nu_3^2 + \nu_4^2) = 2(-gu_3 + h)$$

The solutions of the differential equations generated by the vector field $\mathbf{v}$ in this case are

$$
\begin{cases}
u_1 = r\cos\alpha \\
u_2 = r\sin\alpha \\
u_3 = u_3^0 + \dfrac{g}{2g_3(r)}t - \dfrac{g}{4g_3^2(r)}\sin 2\alpha - -\dfrac{\sqrt{2g}C}{g_3(r)}\cos\alpha \\
u_4 = -h + \dfrac{r^2 g_3^2(r)}{2g} + \Big(\dfrac{\sqrt{g}}{\sqrt{2}g_3(r)}\sin\alpha + C\Big)^2 \\
\alpha = \alpha_0 + g_3(r)t \\
r = \sqrt{u_1^2 + u_2^2}
\end{cases},
$$

where C, r, α_0, u_3^0, h, are arbitrary constants, g_3 is an arbitrary on r function.

To compare with the solutions obtained from the classical approach , firstly we determine the equations of motion obtained from the d'Alembert-lagrange principle are the following

$$
\begin{cases}
\ddot{u_1} = \nu_1 u_1 \\
\ddot{u_2} = \nu_1 u_2 \\
\ddot{u_3} = -g + \nu_2 u_1 \\
\ddot{u_4} = -\nu_2 u_2
\end{cases}
$$

where ν_1, ν_2 are the Lagrangian multipliers.

By derivation the equations generated by the vector field $\mathbf{v}$ we easily obtain

$$\nu_3 = g_3(u_1^2+u_2^2), \quad \nu_1 = -g_3^2(u_1^2+u_2^2), \quad \nu_2 = \nu_3\nu_4 + \frac{gu_1}{u_1^2+u_2^2}$$

and

$$\frac{d}{dt}\nu_4 = -\frac{gu_2}{u_1^2+u_2^2}, \tag{4.2}$$

On the other hand from the condition (4.1) we deduce that $\nu_4 = \nu_4(u_1^2+u_2^2, u_3, u_4)$.

Clearly, the function ν_4 :

$$\nu_4 = \sqrt{\frac{2(-gu_3+h)}{(u_1^2+u_2^2)} - g_3^2(u_1^2+u_2^2)}$$
$$\nu_4 = C + \frac{gu_1}{\nu_3}, \quad C = const.$$

satisfies the above differential equation (4.2). The cartesian approach give as a solution only the first case. In [Gantmacher] the author give as a solution the second case, which produce the potential function

$$U = \frac{1}{2}(C + \frac{gu_1}{\nu_3})^2 + \nu_3^2(u_1^2+u_2^2)) - h$$

which does not correspond to the study case.

5. CARTESIAN APPROACH FOR NON-HOLONOMIC SYSTEM WITH FIVE DEGREE OF FREEDOM AND TWO CONSTRAINTS

This case we shall illustrate in one of the interesting non-holonomic mechanical system: the rattleback.

THE RATTLEBACK

The rattleback's amazing mechanical behaviour is a convex asymmetric rigid body rolling without sliding on a horizontal plane. It is known for its ability to spin in one direction and to resist spinning in the opposite direction for some parameters values, and for others values to exhibit multiple reversals. Basic references on the rattleback are [Wal, Mar, Kar, Bor,Tsyg].

Introduce the Euler angles ψ, ϕ, θ using the principal axis body frame relative to an inertial reference frame. These angles together with two horizontal coordinates x, y of the center of mass are coordinates in the configuration space $\mathcal{Q} = SO(3)\times\mathrm{R}^2$ of the rattleback.

The Lagrangian of the rattleback is computed to be

$$\begin{aligned}
L =&\frac{1}{2}(I_1\cos^2\psi + I_2\sin^2\psi + m(\Gamma_1\cos\theta - \zeta\sin\theta)^2)\dot\theta^2\\
&\frac{1}{2}(I_1\sin^2\psi + I_2\cos^2\psi)\sin^2\theta) + I_3\cos^2\theta)\dot\phi^2\\
&+\frac{1}{2}(I_3 + m\Gamma_2^2\sin^2\theta)\dot\psi^2 + \frac{m}{2}(\dot x^2 + \dot y^2)\\
&+ m(\Gamma_1\cos\theta - \zeta\sin\theta)\Gamma_2\sin\theta\dot\theta\dot\psi + (I_1 - I_2)\sin\theta\sin\psi\cos\psi\dot\theta\dot\phi\\
&C\cos\theta\dot\phi\dot\psi + mg(\Gamma_1\sin\theta + \zeta\cos\theta)
\end{aligned}$$

where I_1, I_2, I_3 are the principal moments of inertia of the body, m is the total mass of the body,

$$\Gamma_1 = \xi\sin\psi + \eta\cos\psi, \quad \Gamma_2 = \xi\cos\psi - \eta\sin\psi$$

$(\xi(\theta,\psi)$, $\eta(\theta,\psi)$, $\zeta(\theta,\psi))$ are the coordinates of the point of contact relative to the body frame.

The shape of the body is encoded by the functions ξ, η and ζ. The constraints are

$$\begin{cases} \dot x - \alpha_1\dot\theta - \alpha_2\dot\psi - \alpha_3\dot\phi = 0\\ \dot y - \beta_1\dot\theta - \beta_2\dot\psi - \beta_3\dot\phi = 0\end{cases} \tag{5.1}$$

where

$$\begin{cases}
\alpha_1 = -(\Gamma_1\sin\theta + \zeta\cos\theta)\sin\phi,\\
\alpha_2 = \Gamma_2\cos\theta\sin\phi + \Gamma_1\cos\phi,\\
\alpha_3 = \Gamma_2\sin\phi + (\Gamma_1\cos\theta - \zeta\sin\theta)\cos\phi,\\
\beta_k = -\dfrac{\partial\alpha_k}{\partial\phi}, \quad k = 1,2,3
\end{cases}$$

It is evident that the rattleback equations of motion in this particular case formally contain the equations of the heavy rigid body in the singular case

$$m \to 0, \quad mg \to l, \quad l \neq 0$$

To determine the Cartesian approach of the Rattleback we first determine the vector field $\mathbf{v}$.

The 1-forms Ω_j, $j = 1, ..., 5$ in this case are the following

$$\begin{cases}
\Omega_1 = dx - \alpha_1 d\theta - \alpha_2 d\psi - \alpha_3 d\phi,\\
\Omega_2 = dy - \beta_1 d\theta - \beta_2 d\psi - \beta_3 d\phi,\\
\Omega_3 = d\theta, \quad \Omega_4 = d\psi, \quad \Omega_5 = d\phi
\end{cases}$$

Hence $\Upsilon = 1$ and

$$\begin{cases}
\mathbf{v} = \lambda_3 X_3 + \lambda_4 X_4 + \lambda_5 X_5\\
X_3 = \alpha_1\partial_x + \beta_1\partial_y + \partial_\theta\\
X_4 = \alpha_2\partial_x + \beta_2\partial_y + \partial_\psi\\
X_5 = \alpha_3\partial_x + \beta_3\partial_y + \partial_\phi
\end{cases} \tag{5.2.}$$

We now proceed to the consideration of the particular case for which ξ, η and ζ are constants. It is easy to show that under this consideration the vector field X_1, X_2, X_3 generated a three dimensional Abelian Lie algebra.

Let $(x^1,\, x^2,\, x^3,\, x^4,\, x^5)$ be a new set of variables derived from $x,$, y, θ, ψ, ϕ by the transformation

$$\begin{cases} \psi = x^1 \\ \phi = x^2, \\ \theta = x^3 \\ y + \zeta \sin\theta\cos\phi - \Gamma_1 \cos\theta \sin\phi - \Gamma_2 \cos\phi = x^4 \\ x + \zeta \sin\theta \sin\phi + \Gamma_1 \cos\theta\cos\phi - \Gamma_2 \sin\phi = x^5 \end{cases}$$

The vector field **v** and the constraints on account of this change, take respectively the form

$$\mathbf{v} = a(x^1, .., x^5))\partial_{x^1} + b(x^1, ..., x^5)\partial_{x^2} + c(x^1, ..., x^5)\partial_{x^3}$$

(5.3) $$\begin{cases} \dot{x}^4 = 0, \\ \dot{x}^5 = 0 \end{cases},$$

By the above transformation the Lagrangian function L is changed into a new Lagrangian

$$\check{L} = \frac{1}{2}\sum_{j,k=1}^{5} G_{jk}(x)\dot{x}^j\dot{x}^k + mg(\Gamma_1 \sin x^3 + \zeta \cos x^3),$$

where $G = (G_{jk})$ is the Riemann metric which is easy to calculate.

We shall now determine the Cartesian approach under the given conditions.

Proposition 5.1

The vector field **v** *given by the formula (4.2) is a Kummer vector field.*

Proof.

In fact, by considering that in this case the 1-form associated to the vector field **v** is the following

(5.4) $$\begin{cases} \sigma = p_1 dx^1 + p_2 dx^2 + p_3 dx^3 \\ p_k = \sum_{j=1}^{5} G_{jk}(x)\mathbf{v}(x^j), \quad k = 1, 2, .., 5 \end{cases}$$

then

$$\imath_{\mathbf{v}} d\sigma = \sum_{j=1}^{5} \Lambda_j dx^j$$

$$\begin{cases} \Lambda_1 = (\frac{\partial p_1}{\partial x^2} - \frac{\partial p_2}{\partial x^1})b + (\frac{\partial p_1}{\partial x^3} - \frac{\partial p_3}{\partial x^1})c \\ \Lambda_2 = (\frac{\partial p_2}{\partial x^3} - \frac{\partial p_3}{\partial x^2})c + (\frac{\partial p_2}{\partial x^1} - \frac{\partial p_1}{\partial x^2})a \\ \Lambda_3 = (\frac{\partial p_3}{\partial x^2} - \frac{\partial p_2}{\partial x^3})b + (\frac{\partial p_3}{\partial x^1} - \frac{\partial p_1}{\partial x^3})a \\ \Lambda_4 = -\frac{\partial p_1}{\partial x^4}a - \frac{\partial p_2}{\partial x^4}b - \frac{\partial p_3}{\partial x^4}c \\ \Lambda_5 = -\frac{\partial p_1}{\partial x^5}a - \frac{\partial p_2}{\partial x^5}b - \frac{\partial p_3}{\partial x^5}c \end{cases}$$

Let $\mathbf{v}(x)$ and $rot\mathbf{v}(x)$ are the following vectors

$$\mathbf{v}(x) = (a,\, b,\, c)$$

$$rot\mathbf{v} = \frac{1}{\sqrt{detG}}(\frac{\partial p_3}{\partial x^2} - \frac{\partial p_2}{\partial x^3}, \frac{\partial p_1}{\partial x^3} - \frac{\partial p_3}{\partial x^1}, \frac{\partial p_2}{\partial x^1} - \frac{\partial p_1}{\partial x^2})$$

We have therefore that the equations (2.5),(2.6) take the form respectively

(5.5)
$$\begin{cases} \dot{x}^1 = a(x^1, x^2, x^3, C_4, C_5) \\ \dot{x}^2 = b(x^1, x^2, x^3, C_4, C_5) \\ \dot{x}^3 = c(x^1, x^2, x^3, C_4, C_5) \end{cases}$$

(5.6)
$$[\mathbf{v} \times rot\mathbf{v}(x)] = 0,$$

hence the constructed vector field is a Kummer vector field.

For the general case, i.e., when the ξ, η and ζ are functions on the variables θ and ψ the Cartesian approach produce the following equations

$$\dot{x}^k = \mathbf{v}(x^k), \quad k = 1, 2, .., 5$$

$$\begin{cases} \sum_{j=1}^{5} \Big(\frac{\partial p_1}{\partial x^j} - \frac{\partial p_j}{\partial x^1} + \alpha_2(\frac{\partial p_4}{\partial x^j} - \frac{\partial p_j}{\partial x^4}) + \beta_2(\frac{\partial p_5}{\partial x^j} - \frac{\partial p_j}{\partial x^5})\Big)\mathbf{v}(x^j) = 0 \\ \sum_{j=1}^{5} \Big(\frac{\partial p_2}{\partial x^j} - \frac{\partial p_j}{\partial x^2} + \alpha_3(\frac{\partial p_4}{\partial x^j} - \frac{\partial p_j}{\partial x^4}) + \beta_3(\frac{\partial p_5}{\partial x^j} - \frac{\partial p_j}{\partial x^5})\Big)\mathbf{v}(x^j) = 0 \\ \sum_{j=1}^{5} \Big(\frac{\partial p_3}{\partial x^j} - \frac{\partial p_j}{\partial x^1} + \alpha_1(\frac{\partial p_4}{\partial x^j} - \frac{\partial p_j}{\partial x^4}) + \beta_1(\frac{\partial p_5}{\partial x^j} - \frac{\partial p_j}{\partial x^5})\Big)\mathbf{v}(x^j) = 0 \end{cases}$$

where

$$\begin{cases} \psi = x^1, & \phi = x^2, \\ \theta = x^3, & y = x^4, \quad x = x^5 \end{cases}$$

and $\mathbf{v}$ is given by the formula (5.2).

6. THE CARTESIAN APPROACH IN THE CASE II

For the case II the vector field $\mathbf{v} \equiv \mathbf{v}_w$ is the following

(6.1)

$$\mathbf{v}_w = \frac{det}{\Upsilon}\begin{pmatrix} 2da_1(\partial_1)+2w & da_1(\partial_2)+da_2(\partial_1) & \dots & da_1(\partial_N)+da_N(\partial_1) & a_1 \\ da_1(\partial_2)+da_2(\partial_1) & 2da_2(\partial_2)+2w & \dots & da_2(\partial_N)+da_N(\partial_2) & a_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ da_M(\partial_1)+da_1(\partial_M) & da_M(\partial_2)+da_2(\partial_M) & \dots & da_M(\partial_N)+da_N(\partial_M) & a_M \\ \vdots & \vdots & \dots & \vdots & \vdots \\ da_N(\partial_1)+da_1(\partial_N) & da_N(\partial_2)+da_2(\partial_N) & \dots & 2da_N(\partial_N)+2w & a_N \\ \partial_1 & \partial_2 & \dots & \partial_N & 0 \end{pmatrix},$$

By introducing the matrix

$$B+wI = \begin{pmatrix} 2da_1(\partial_1)+2w & da_1(\partial_2)+da_2(\partial_1) & \dots & da_1(\partial_N)+da_N(\partial_1) \\ da_1(\partial_2)+da_2(\partial_1) & 2da_2(\partial_2)+2w & \dots & da_2(\partial_N)+da_N(\partial_2) \\ \vdots & \vdots & \dots & \vdots \\ da_M(\partial_1)+da_1(\partial_M) & da_M(\partial_2)+da_2(\partial_M) & \dots & da_M(\partial_N)+da_N(\partial_M) \\ \vdots & \vdots & \dots & \vdots \\ da_N(\partial_1)+da_1(\partial_N) & da_N(\partial_2)+da_2(\partial_N) & \dots & 2da_N(\partial_N)+2w \end{pmatrix},$$

we obtain the equivalent representation for $\mathbf{v}$

(6.2)
$$\mathbf{v} = \frac{\sum_{k=1}^{N} w^{N-k} X_k}{det(B+Iw)}$$

(6.3)
$$\begin{cases} X_1 = \sum_{j=1}^{N} a_j(x)\partial_j \\ X_2 = \sum_{j=1}^{N}(4da_j(\partial_j)X_1 - \partial_j(\sum_{k=1}^{N} a_k^2(x))\partial_j) - 2\sum_{j,k=1}^{N} a_j(x)\partial_j a_k(x)\,\partial_k \\ \dots\dots\dots\dots\dots\dots\dots \\ X_N = \mathbf{v}_w|_{w=0} \end{cases}$$

or, what is the same,

$$\mathbf{v}_w = (\mathbf{a}, (B+Iw)^{-1}\partial\,)$$

where $\mathbf{a} = (a_1, a_2, .., a_N)$ and $\partial = (\partial_1, \partial_2, ..., \partial_N.)$

Corollary 6.1

Let us denote by F the following function

$$F(w) = \Omega(\mathbf{v}_w) = \frac{\sum_{k=1}^{N} w^{N-k}\Omega(X_k)}{det(B + Iw)}$$

where

$$\Omega = \sum_{j=1}^{N} a_j(x)dx^j$$

then

1. The all zeros of F are real.

2. Let $w_1, w_2, ..., w_{N-1}$ the different real zeros of the function F then the following relations hold

$$\begin{cases} (-)^{N-1}\sum_{j=1}^{N-1} w_j = \frac{\Omega(X_2)}{\Omega(X_1)} \\ \dots\dots\dots\dots\dots \\ (-1)^N \prod_{j=1}^{N-1} w_j = \frac{\Omega(X_N)}{\Omega(X_1)} \end{cases} \tag{6.4}$$

In particular, for $N = 3$ and $\Omega = df$ we have

$$\begin{cases} w_1 + w_2 = df(X_1)H \\ w_1 w_2 = (df(X_1))^{\frac{3}{2}} K \end{cases}$$

where H and K are the middle and Gaussian curvature of the surface $f(x, y, z) = c$.

Corollary 6.2

Let $w_1, w_2, ..., w_{N-1}$ the different real zeros of the function F and l et we denote by $\mathbf{v}_j$, $j = 1, 2, ..., N-1$ the vector field:

$$v_j = v_w|_{w=w_j}, \quad j = 1, 2, ..., N-1$$

then:

1. $v_1, v_2, ..., v_{N-1}$ are orthogonal.

2 The most general characteristic element of the -form Ω admits the representation

$$\tilde{\mathbf{v}} = \sum_{j=1}^{N-1} \kappa_j \mathbf{v}_j = \sum_{j=1}^{N-1}\sum_{k=1}^{N} \frac{w_j^{N-k}\kappa_j X_k}{det(B + Iw_j)} \equiv \sum_{k=1}^{N} \tilde{\lambda}_k X_k, \tag{6.5}$$

where

$$\tilde{\lambda}_k = \sum_{j=1}^{N-1} \frac{w_j^{N-k}\kappa_j}{det(B+Iw_j)} \quad k = 1, 2, .., N.$$

The proof of the first assertion is analogously to the proof of the proposition 6.2.

To determine the Cartessian approach in this case we need to construct the 1-form σ associated to the vector field $\tilde{\mathbf{v}}$. It is easy to prove that

$$\sigma = \sum_{j=1}^{N} \tilde{\lambda}_k \mu_k,$$

where $\mu_k, \quad k = 1, 2, .., N$ are 1-form associated to the vector fields $X_1, X_2, ..X_N$. Clearly,

$$\mu_1 = \Omega = \sum_{j=1}^{N} a_j(x) dx^j$$

As a consequence

$$\imath_{\tilde{\mathbf{v}}} d\sigma = \Lambda_1 \Omega + \sum_{j=2}^{N} \Lambda_j \mu_j \tag{6.6}$$

Proposition 6.1

The Cartessian and Lagrangian approach for a mechanical system with kinetic energy

$$T = \frac{1}{2} \sum_{j,k=1}^{N} G_{jk} \dot{x}^j \dot{x}^k$$

with the constraints generated by the 1-form Ω produce the following equations respectively

$$\dot{x} = \sum_{k=1}^{N} \tilde{\lambda}_k X_k(x) \tag{6.7}$$

with the conditions

$$\begin{cases} \Lambda_j = 0, \quad j = 2, 3, ..., N \\ det(B + Iw_k) \neq 0, \quad k = 1, 2, .., N-1 \end{cases} \tag{6.8}$$

and

$$E_k(T) = \partial_j \frac{1}{2} ||\mathbf{v}||^2 + \Lambda_1 a_j, \quad j = 1, 2, .., N$$

which can be interpreted as equations of motion under the active forces with potential $2(U+h) = ||\tilde{\mathbf{v}}|^2$ and reactive forces with components

$$(\Lambda_1 a_1, \; \Lambda_1 a_2, ..., \Lambda_1 a_N)$$

generated by the constraint

$$\sum_{j=1}^{N} a_j \dot{x}^j = 0.$$

Now we shall study the case when

$$\mathcal{Q} = \mathrm{R}^N, \quad G = I = diag(1, ..., 1), \quad \Omega = df$$

In this case the matrix $B + wI$ is symmetrically, as a consequence the all zeros the function F :

$$F(w) = (gradf, (B + Iw)^{-1} gradf)$$

are real.

Proposition 6.2

Let $\mathbf{v}_j$ and $\mathbf{v}_k$ the vector field correspond to the roots w_j and w_k respectively. Then they are orthogonal.

In fact, the zeros of F coincide with the values of the Lagrangian multipliers in the following problem:

Determine

$$Extremum(\frac{1}{2}\sum_{j,k=1}^{N} \frac{\partial^2 f}{\partial x^j \partial x^k} \tau^j \tau^k)$$

under the conditions

$$\begin{cases} \sum_{j=1}^{N} \tau_j^2 - 2 = 0 \\ df(\tau) = 0 \end{cases}$$

As well known the solution of this problem require to solve the system

$$\begin{cases} \sum_{j,k=1}^{N} \frac{\partial^2 f}{\partial x^j \partial x^k} \tau^j + \nu \frac{\partial f}{\partial x^j} + w\tau_j = 0 \\ \sum_{j=1}^{N} \tau_j^2 - 2 = 0 \\ df(\tau) = 0 \end{cases} \tag{6.9}$$

This system can be represented in matrix form as follows

$$\mathcal{R}\vec{b} = 0$$

where

$$\mathcal{R} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} + w & \frac{\partial^2 f}{\partial x^1 \partial x^2} & \dots & \frac{\partial^2 f}{\partial x^1 \partial x^N} & \frac{\partial f}{\partial x^1} \\ \frac{\partial^2 f}{\partial x^1 \partial x^2} & \frac{\partial^2 f}{\partial x^2 \partial x^2} + w & \dots & \frac{\partial^2 f}{\partial x^2 \partial x^N} & \\ \vdots & \vdots & \dots & \vdots & \\ \frac{\partial^2 f}{\partial x^1 \partial x^N} & \frac{\partial^2 f}{\partial x^2 \partial x^N} & \dots & \frac{\partial^2 f}{\partial x^N \partial x^N} + w & \frac{\partial f}{\partial x^N} \\ \frac{\partial f}{\partial x^1} & \frac{\partial f}{\partial x^2} & \dots & \frac{\partial f}{\partial x^N} & 0 \end{pmatrix} \tag{6.10}$$

$$\vec{b} = col(\tau_1, \tau_2, ..., \tau_N, \nu)$$

Clearly, $det\mathcal{R} = det(B + Iw)F(w)$

By considering that the vector $\vec{b}$ is nonzero vector, hence we obtain that the given system holds if and only if

$$det\mathcal{R} = det(B + Iw)F(w) = 0$$

If we denote by τ_α and τ_β the solutions of (6.8) for the values w_α and w_β of the lagrangian multiplier w, then after some calculations we can prove that

$$(w_\alpha - w_\beta)\sum_{j=1}^{N} \tau_\alpha^j \tau_\beta^j = 0$$

hence we obtain the orthogonality of the given vector fields.

The vector field τ_α can be determine as follow

$$\tau_\alpha = \frac{\mathbf{v}_\alpha}{||\mathbf{v}_\alpha||}$$

CARTESSIAN AND LAGRANGIAN APPROACH FOR A PARTICLE ON THE ELIPSOIDE

Now we shall study the particular case when the the function f :

$$f = \sum_{j=1}^{N} a_j (x^j)^2, \quad a_1 < a_2 < .... < a_N$$

The vector field $\mathbf{v}_w$ in this case can be represented as follows

$$\mathbf{v}_w = \frac{\sum_{k=1}^{N} w^{N-k} X_k}{\prod_{j=1}^{N}(w + a_j)} \tag{6.11}$$

where

$$\begin{cases} X_1 = \mathbf{grad} f_1, \quad 2f_1 = \sum\limits_{j=1}^{N} a_j (x^j)^2 \\ X_2 = \mathbf{grad} f_2, \quad 2f_2 = \sum\limits_{j=1}^{N} \sum\limits_{k_1 \neq j} a_{k_1} a_j (x^j)^2 \\ X_3 = \mathbf{grad} f_3, \quad 2f_3 = \sum\limits_{j=1}^{N} \sum\limits_{k_1 \neq k_2 \neq j} a_{k_1} a_{k_2} a_j (x^j)^2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ X_N = \mathbf{grad} f_N, \quad 2f_N = \prod\limits_{j=1}^{N} a_j \sum\limits_{k=1}^{N} (x^k)^2 \end{cases}$$

Corollary

The set of vector field X_1, X_2, ..., X_N formed an Abelian Lie algebra.

Moreover, from the above formula we easily obtain

$$\mathbf{v}_w = -\sum_{k=1}^{N} \frac{a_k x^k}{a_k + w} \partial_k \tag{6.11}$$

The function F in this case takes the form

$$F(w) = df(\mathbf{v}_w) = -\sum_{k=1}^{N} \frac{(a_k x^k)^2}{a_k + w} \tag{6.12}$$

Proposition 6.3

The all zeros of F are simple.

In fact, by considering that the matrix $\mathcal{R}$ is symmetrically we obtain that the zeros of F are real. on the other hand from the equality

$$\frac{dF}{dw} = \sum_{k=1}^{N} (\frac{a_k x^k}{a_k + w})^2, \quad \frac{dF}{dw}|_{w=w_j} > 0, \, j = 1, 2, .., N-1,$$

we deduced that the zeros are simply.

By considering that for F is valid the following representation

$$F(w) = \frac{\prod_{j=1}^{N-1} (w - w_j)}{\prod_{k=1}^{N} (w + a_k)}$$

and by using the residuos theorem we have

$$a_k^2 (x^k)^2 = res_{w=-a_k} F(w) = df(X_1) \frac{\prod_{j=1}^{N-1} (w_j + a_k)}{\prod_{n \neq k} (a_n - a_k)}$$

Definition Let $0 < a_1 < a_2 < .... < a_N$ be distinct positive numbers. For each $x = (x_1, x_2, ..., x_N) \in \mathrm{R}^N$ the equation

$$\sum_{j=1}^{N} \frac{(a_k x^k)^2}{a_k + w} = 1 \tag{6.13}$$

defines n real numbers α_1, $\alpha_2, ..., \alpha_N$ which separate a_1, a_2, ..., a_N.

The number α_1, $\alpha_2, ..., \alpha_N$ serve sa curvilinear coordinates in R^N and are called Jacobi's elliptic coordinates.

Clearly, the equation of the elipsoide

$$\sum_{j=1}^{N} a_k (x^k)^2 = 1$$

in this coordinates is $\alpha_N = 0$. Below we consider that this equation holds.

By introducing the function $G(w) = F(w) - 1$ we obtain

$$G(w) = \frac{-w^N + \sum_{j=1}^{N}(a_j - df(X_1)w^{N-1} ..... + (-1)^{N-1} \prod_{j=1}^{N} a_j - df(X_N))}{\prod_{k=1}^{N}(w + a_k)} = \frac{B(w)}{A(w)}$$

where

$$B(w) = \prod_{j=1}^{N-1} w(w - \alpha_j), \quad A(w) = \prod_{k=1}^{N}(w + a_k)$$

Hence we deduce

$$(6.14) \qquad (a_k x^k)^2 = res_{w=-a_k} G(w) = -\frac{a_k \prod_{k=1}^{N-1}(\alpha_j + a_k)}{\prod_{j \neq k}(a_k - a_j)}$$

The formula

$$(x^k)^2 = \frac{B(a_k)}{A'(a_k)} = -\frac{\prod_{k=1}^{N-1}(\alpha_j + a_k)}{a_k \prod_{j \neq k}(a_k - a_j)}$$

expressed the coordinates on the elipsoide through the eliptic coordinates $\alpha_1, \alpha_2, .., \alpha_{N-1}$.

From (6.14) we give

$$(6.15) \qquad \begin{cases} df(X_1) = \sum_{j=1}^{N} a_j + (-1)^{N-1} \sum_{j=1}^{N} \alpha_j, \quad \alpha_N = 0 \\ df(X_2) = \sum_{k_1 \neq k_2}^{N} a_{k_1} a_{k_2} + \sum_{k_1 \neq k_2}^{N} \alpha_{k_1} \alpha_{k_2}, \quad \alpha_N = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ df(X_N) = (-1)^N \prod_{j=1}^{N} a_j + (-1)^N \prod_{j=1}^{N} \alpha_j, \quad \alpha_N = 0 \Longleftrightarrow \sum_{j=1}^{N} a_k (xk)^2 = 1 \end{cases}$$

Proposition 6.4

The most general characteristic element $\mathbf{v}$ *of the 1-form*

$$\Omega = \sum_{j=1}^{N} a_k x^k dx^k$$

admits the following equivalent representations

$$(6.16) \qquad \mathbf{v} = -\sum_{j=1}^{N-1} \sum_{k=1}^{N} \frac{\kappa_j a_k x^k}{a_k + w_j} \partial_k$$

$$(6.17)\quad \begin{cases} \mathbf{v} = \sum_{j=1}^{N} \tilde{\lambda}_j X_j = \sum_{j=1}^{N} \tilde{\lambda}_j \mathbf{grad} f_j \\ \tilde{\lambda}_n = \sum_{j=1}^{N-1} \frac{w_j^{N-n} \kappa_j}{\prod_{j=1}^{N} (w_j + a_n)} \end{cases}$$

$$(6.18)\quad \begin{cases} \mathbf{v} = \frac{\sum_{n=2}^{N} (a_k)^{N-n} \psi_n}{a_k x^k \prod_{s \neq k} (a_s - a_k)} \partial_k \\ \psi_2 = \sum_{j=1}^{N-1} \tilde{\kappa}_j, \quad \tilde{\kappa}_j = df(X_1) \kappa_j \\ \psi_3 = \sum_{j=1}^{N-1} \sum_{n \neq j} w_n \tilde{\kappa}_j \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \psi_N = \sum_{j=1}^{N-1} \prod_{n \neq j} w_n \tilde{\kappa}_j \end{cases}$$

From these relations we obtain

$$dg_j(\mathbf{v}) = 2\psi_j, \quad g_j(x) = \frac{1}{2} \sum_{k=1}^{N} a_k^j x_k^2, \quad j = 1, 2, ..N-1$$

and finally in elliptic coordinates

$$(6.19)\quad \mathbf{v} = \sum_{n=1}^{N} \frac{(-1)^n (\alpha_j)^{N-n} \psi_n}{\prod_{s \neq j} (\alpha_j - \alpha_s)} \partial_{\alpha_j}$$

To determine the Cartesian approach for the study case we determine the 1-form σ associated with the vector field $\mathbf{v}$.

Clearly, if $\mathbf{v}$ admits the representation (6.17) then

$$\sigma = \sum_{j=1}^{N} \tilde{\lambda}_j df_j,$$

Hence,

$$\imath_{\mathbf{v}} d\sigma = \Lambda_1 df_1, \quad \Lambda_1 = \sum_{j=2}^{N} (\partial_{f_j} \tilde{\lambda}_1 - \partial_{f_1} \tilde{\lambda}_j) df_j(\mathbf{v})$$

if and only if

$$\tilde{\lambda}_j = \partial_{f_j} \Phi(f_1, f_2, ..., f_N), \quad j = 2, 3, .., N$$

Hence, the Cartesian and lagrangian approach for the study case are the following

$$\begin{cases} \dot{x}^k = \sum_{j=1}^{N-1} \dfrac{\kappa_j a_k x^k}{a_k + w_j} \\ \begin{cases} \sum_{j=1}^{N-1} \dfrac{w_j^{N-n}\kappa_j}{det(B+Iw_j)} = \partial_{f_j}\Phi(f_1, f_2, ..., f_N), \quad 2,3,..,N \\ \sum_{j=1}^{N-1} \dfrac{w_j^{N-1}\kappa_j}{det(B+Iw_j)} = \tilde{\lambda}_1 \end{cases} \end{cases}$$

$$\begin{cases} \ddot{x}^k = \partial_k \dfrac{1}{2}||\mathbf{v}||^2 + \Lambda_1 a_k x^k, \quad k = 1, 2, .., N \\ 2||\mathbf{v}||^2 = \sum_{j=1}^{N} (\tilde{\lambda}_j)^2 ||\mathbf{grad} f_j||^2 \\ \Lambda_1 = \sum_{j=2}^{N} (\partial_{f_j}\tilde{\lambda}_1 - \partial_{f_1}\tilde{\lambda}_j) df_j(\mathbf{v})) \end{cases}$$

If the vector field we determine by the formula (6.19) then by considering that in this case

$$\sum_{j=1}^{N} \dot{x}_j^2 = \sum_{j=1}^{N-1} F^{'}(\alpha_j)\dot{\alpha}_j^2$$

i.e.,

$$G = diag(F^{'}(\alpha_1),\ F^{'}(\alpha_2), ..., F^{'}(\alpha_{N-1}))$$

hence the 1-form σ in elliptic coordinates takes the form

$$\sigma = \sum_{j=1}^{N-1} F^{'}(\alpha_j)\dot{\alpha}_j d\alpha_j = \sum_{n=2}^{N} \psi_j d\varrho_j$$

where

$$d\varrho_n = \sum_{j=1}^{N-1} \frac{(-1)^n(\alpha_j)^{N-n} d\alpha_j}{\prod_{s=1}^{N}(\alpha_j - a_s)}, \quad n = 2, 3, ..., N$$

we shall study the case when

$$\sigma = dS,$$

which we obtain if the arbitrary functions ψ_j, $j = 2, 3, .., N$ are such that

$$\psi_n = \partial_{\varrho_n} S = \sum_{j=1}^{N-1} F^{'}(\alpha_j)\alpha_j)^{N-n}\partial_{\alpha_j} S, \quad n = 2, 3, .., N$$

Corollary 6.3

If

$$\psi_n = \psi_n(\varrho_n)$$

then the problem of motion of a point on the elipsoide is integrable by method of separation of variables.

In fact in this case the function S admits the representation

$$S = \sum_{j=2}^{N} S_j(\varrho_j), \quad S_j = \int \psi_(\varrho_j) d\varrho_j, \quad j = 2, .., N$$

Clearly, under this conditions the Cartesian and Lagrangian approach produce the following equations respectively

$$\dot{\alpha}_j = \frac{\partial_{\alpha_j} S}{F'(u_j)}$$

$$\begin{cases} E_k(L) = 0, \quad k = 1, .., N-1\,, \\ L = \frac{1}{2}\sum_{j=1}^{N-1}(F^{'}(u_j)(\dot{\alpha}_j)^2 - \frac{(\partial_{\alpha_j} S)^2}{F'(u_j)}) \end{cases}$$

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