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Algebraic Equations of All Involucre Conics in the Configuration of the c-Inscribed Equilateral Triangles of a Triangle

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Abstract. Let $\triangle ABC$ be a triangle with side length $c = AB$; here we present the determination of the existence and quantity m of the c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ (i.e. $\mathbb{T}_j = \triangle A_jB_jC_j$ with $A_j \in \overleftrightarrow{BC}$, $B_j \in \overleftrightarrow{CA}$, $C_j \in \overleftrightarrow{AB}$, $c = A_jB_j$) of $\triangle ABC$ in function of the position of vertex C respect to a separatrix parabola $\mathcal{P}_i$, and from an algebraic point of view. We give the algebraic equations of all involucre conics –circles $\mathbb{N}_o$, $\mathbb{N}_i$; parabola $\mathcal{P}_i$; ellipses $\mathcal{H}_i$, $\mathcal{H}_o$– in the configuration.

1. Introduction

Many configurations linking conics and equilateral triangles with the triangle have been described by different geometers in the past; here we give a new one. Let $\triangle ABC$ be a triangle with side length $c = AB$; in this work we want to present the determination of the existence and quantity of the c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ (i.e. $\mathbb{T}_j = \triangle A_jB_jC_j$ with $A_j \in \overleftrightarrow{BC}$, $B_j \in \overleftrightarrow{CA}$, $C_j \in \overleftrightarrow{AB}$, $c = A_jB_j$) of $\triangle ABC$ in function of the position of vertex C respect to a separatrix parabola $\mathcal{P}_i$, from an algebraic point of view. We give the algebraic equations of all involucre conics –circles $\mathbb{N}_o$, $\mathbb{N}_i$; parabola $\mathcal{P}_i$; ellipses $\mathcal{H}_i$, $\mathcal{H}_o$– in the configuration.

Readers can find the construction of the c-inscribed equilateral triangles [3]. And from the kinematic point of view we are considering a well known result of planar kinematics: we consider the motion of the point X of an equilateral triangle $\triangle PQX$, where P and Q slide along straight (non-parallel) lines. It is well known that, in the general case, the trajectory of X is an ellipse (for each of the two possible orientations of $\triangle PQX$). Therefore, we consider nothing else than a special case of the well known elliptic motion or Cardan motion [1], [2]. Nevertheless, in this work, through long but straightforward calculations, we present not the well known kinematic point of view, but the algebraic equations of the special case of all the conics which are linked with the c-equilateral triangles which are sliding on a triangle $\triangle ABC$. More precisely, let $\triangle ABC$ be a triangle with side length $c = AB$, let be their equilateral triangles, of side length c, which are sliding on the straight lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$. In the next section we present the algebraic equations

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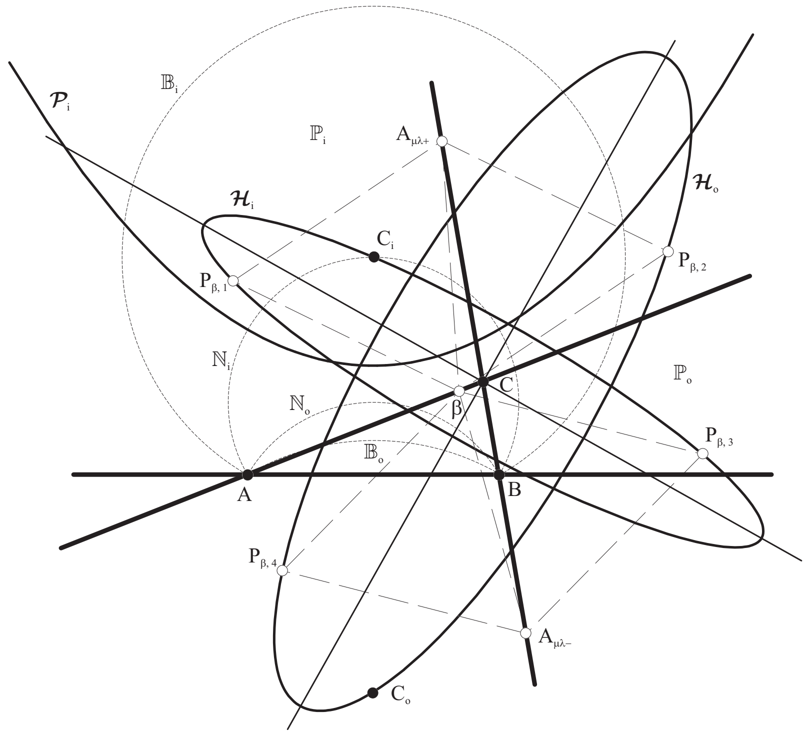

Figure 1. The conics linked with the c-equilateral triangles which are sliding on a triangle.

of all the conics which are linked with these c-equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ (see Figure 1). And with these configuration equations, we present the determination of the existence and quantity m of the c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ of $\triangle ABC$ in function of the position of vertex C respect to a separatrix parabola $\mathcal{P}_i$, from an algebraic point of view. (see Figures 2, 3).

Of course, the triangle $\triangle ABC$ has its other two similar configurations for its other two sides $b = AC$, $a = BC$.

1.1. *Elements of the configuration.* In the following we fix, with precision, the notation and the elements involved with the configuration (for the case of the side $\overline{AB}$).

Lemma 1. *Let $\triangle ABC$ be a triangle in the affine euclidean plane $\mathbb{A}^2$, with its length side $c = AB$, and (see Figures 1, 2, 3):*

1.- Let $\{T_{\beta,k} = \triangle P_{\beta,k} A_{\beta,k} \beta\}_{k=1}^{k=4}$ be its four β-sliding equilateral triangles: i.e. β is an arbitrary point with $\beta \in \overleftrightarrow{CA}$, $A_{\beta,k} \in \overleftrightarrow{BC}$, and $T_{\beta,k}$ has its length sides equal to $c = AB$ (see Figure 1, and the proof of Lemma 2) [this is a special case of the well known elliptic motion].

2.- Let $\mathbb{N}_o$ and $\mathbb{N}_i$ be the circumcircle of $\triangle ABC_o$ and $\triangle ABC_i$, the outer equilateral triangle and the inner equilateral triangle of $\overline{AB}$, respectively. Let $\mathbb{B}_o$ and